Gravitational Radiation and Very Long Baseline Interferometry

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Abstract

Gravitational waves affect the observed direction of light from distant sources. At telescopes, this change in direction appears as periodic variations in the apparent positions of these sources on the sky; that is, as proper motion. A wave of a given phase, traveling in a given direction, produces a characteristic pattern of proper motions over the sky. Comparison of observed proper motions with this pattern serves to test for the presence of gravitational waves. A stochastic background of waves induces apparent proper motions with specific statistical properties, and so, may also be sought. In this paper we consider the effects of a cosmological background of gravitational radiation on astrometric observations. We derive an equation for the time delay measured by two antennae observing the same source in an Einstein-de Sitter spacetime containing gravitational radiation. We also show how to obtain similar expressions for curved Friedmann-Robertson-Walker spacetimes.

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1. Introduction

It is commonly agreed that gravitational waves, predicted by Einstein's theory of general relativity (Einstein 1916), must exist in our Universe. To date, however, the only evidence for their presence is that the orbital decay rates of certain binary pulsars appear consistent with the predicted rates of energy loss from gravitational radiation (Taylor 1992; Taylor and Weisberg 1989). Nevertheless, gravitational waves arise generically after the inflationary phase in inflationary cosmologies (Rubakov, Sazhin, and Veryaskin 1982; Fabbri and Pollock 1983; Abbott and Wise 1984) and should be produced in a wide range of physical situations at later times (Thorne 1987; Carr 1980). For these reasons astrophysicists are certain that gravitational wave astronomy, though difficult, will be of enormous value in understanding our Universe.

The effects of gravitational radiation may be divided into two categories. Direct effects physically couple the energy density in the waves to matter, causing, for instance, a bar to resonate. Gravitational waves also affect the propagation of radiation, causing a spacetime containing gravitational waves to look different from one without. The very long wavelength ($\lambda > 10^{-3}$ pc) gravitational radiation which we focus on in this paper is best searched for by examining its effects on the radiation we receive from astrophysical sources. Such indirect effects have been used successfully to constrain the fraction of the energy density in our Universe which can be contained in gravitational radiation of various wavelengths. Among these constraints are $\Omega_{\rm GW} < 10^{-4}$ at $\lambda \approx 1$ pc, from pulsar timing (Romani and Taylor 1983; Taylor 1987); $\Omega_{\rm GW} < 0.04$ at 10 pc $\leq \lambda \leq$ 10 kpc, also from pulsar timing (Taylor and Weisberg 1989); $\Omega_{\rm GW} < 10^{-4}$ at $\lambda \leq 0.1$ kpc if the waves existed during nucleosynthesis, from nucleosynthesis constraints (Carr 1980); $\Omega_{\rm GW} < 10^{-8}$ or 10^{-3} at $\lambda > 1$ Mpc if the waves did or did not exist, respectively, at recombination, from microwave background anisotropy limits (Linder 1988a); and $\Omega_{\rm GW} < 10^{-3}$ for 30 kpc $\leq \lambda \leq$ 300 Mpc, from galaxy-galaxy n-point correlation functions (Linder 1988b).

Recently Eubanks and Matsakis (1994) have reported Very Long Baseline Interferometry (VLBI) measurements that indicate quasars have a definite pattern of apparent motions on the sky with root-mean-square (RMS) angular velocity ~ 20 μas yr^{-1}. The work of Linder (1988b) furnishes an estimate of the constraint we can expect from data of this accuracy. Linder obtains an expression for the mean-square angular deflection of light from cosmological sources induced by gravitational waves. Dividing this expression by the square of the wave period we gain an estimate for the mean-square angular velocity of sources at redshift z in an Einstein-de Sitter spacetime induced by waves with energy density as a fraction of the closure density $\Omega_{\rm GW}$ given by $\langle\omega^2\rangle \approx \left(1+(1+z)^2\right) H_o^2\Omega_{\rm GW}$ where H_o is the Hubble constant at $z=0$. Using $H_o = 100h$ km s^{-1} Mpc^{-1} this tells us (taking $z=1$ as a representative redshift) that the quasar motion data may be expected to either constrain or detect gravitational waves at a level of $\Omega_{\rm GW} = 0.04\mathrm{h}^{-2}$. This is competitive with the pulsar timing limits, and should cover a much larger range of gravitational wave wavelengths.

In order to test the hypothesis that the quasar motions reported by Eubanks and Matsakis (1994) are caused by a cosmologically significant background of gravitational waves we need a theoretical framework suitable for the analysis of the effects of such waves on VLBI measurements of distant sources. In a seminal work on observations in cosmology, Kristian and Sachs (1965) established a number of formulae relating the observed properties of cosmological sources to the physical properties of the spacetime in which they are observed. Their formula for the proper motion distance could be used immediately to analyze the system we are concerned with here except that their work utilizes an expansion in the distance to the source divided by some reasonably defined radius of curvature of the spacetime. For the high redshift quasars of Eubanks and Matsakis (1994) such an expansion is not useful as the quasars are a large fraction of the Hubble distance, and many gravitational wave wavelengths, from us. The work of Linder (1988b) also bears close relationship to the problem under consideration. Linder has obtained the deviation in angle suffered by a light ray in an Einstein-de Sitter spacetime containing gravitational waves. For astrophysical thin lens systems, where the angular deflection of an incident light ray may be considered to occur at a single point, knowledge of the angular deviation is sufficient to determine the apparent position of the source on the observer's sky. This is the content of the well known lens equation. The situation is different, however, when a cosmological background of gravitational waves is effectively acting as the lens. For the wave case, the angular deflection occurs over the entire photon path and there is no obvious *a priori* relationship between the source position on the observer's sky and the purely mathematical, i.e. unobservable, angular deviation.

In this paper we use the perturbative geodesic expansion introduced in Pyne and Birkinshaw (1993) to determine the effects of gravitational radiation on VLBI measurements of distant sources. In Gwinn *et al.* (1995) we use the results of this work to test the hypothesis that the quasar motions reported by Eubanks and Matsakis (1994) are caused by a cosmologically significant background of gravitational waves.

The outline of this paper is as follows. In section 2 we develop a method for analyzing a VLBI experiment in metric perturbed Einstein de-Sitter spacetimes. In section 3 we apply the method to the case of an Einstein-de Sitter spacetime perturbed by a spectrum of cosmological gravitational waves. In section 4 we consider a single, plus-polarized, monochromatic wave and determine the pattern of source proper motions which it produces on the sky. In section 5 we generalize our equations to the curved Friedmann-Robertson-Walker (FRW) spacetimes and show how a physical understanding of the method emerges from an analysis of the Jacobi equation. In section 6 we investigate the consistency of our equations by considering a simple gravitational lens system. In section 7 we present our conclusions.

2. Light Rays in a Perturbed Einstein-de Sitter Spacetime

The results of Pyne and Birkinshaw (1995) allow construction of the paths of light rays through a perturbed Einstein-de Sitter spacetime with only minimal effort. The metric for such a spacetime takes the form

$$d\bar{s}^2 = a^2 \left(-d\eta^2 + dx^2 + dy^2 + dz^2\right) + a^2 h_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

where a is the Friedmann expansion factor. We let $d\bar{s}^2 = a^2 ds^2$. By standard conformal results, light rays in $d\bar{s}^2$ and ds^2 coincide and their (affine) parameterizations are related by $\bar{k}^\mu = a^{-2} k^\mu$. Here, and throughout this paper, Roman letters $i, j, ...$ run over $\{1, 2, 3\}$, while Greek letters $\mu, \nu, ...$ run over $\{0, 1, 2, 3\}$. We use geometrized units, $G = c = 1$. The spacetime metric is taken to have signature $+2$. Our Riemann and Ricci tensor conventions are given by $[\Delta_\alpha, \Delta_\beta] v^\mu = R^\mu{}_{\nu\alpha\beta} v^\nu$ and $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$.

The metric, (1), is of a class of metrics whose radial null geodesics were investigated in Pyne and Birkinshaw (1995). For the specific case of (1), the Einstein-de Sitter background, non-radial null geodesics may be constructed from the results of that paper with almost no effort (this is because the Jacobi and parallel propagators for these geodesics are the same as those used in that work. For the case of the curved FRW backgrounds, the propagators for the radial and the non-radial geodesics are not equivalent). We express this in the form of a

Theorem: Let $ds^{(0)2}$ denote that part of ds^2 independent of the perturbation (i.e. the Minkowski metric). Let $x^{(0)\mu}(\lambda)$ be an affinely parametrized null geodesic of $ds^{(0)2}$ with $k^{(0)\mu}(\lambda) = dx^{(0)\mu}(\lambda)/d\lambda$ such that $k^{(0)0} = 1$. Put

$$f^{(1)\mu} = -\Gamma^{(1)\mu}{}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta}. \tag{2}$$

with $\Gamma^{(1)\mu}{}_{\alpha\beta}$ that part of the Christoffel connection of ds^2 which is linear in the metric perturbation and its first partial derivatives. Then $x^\mu(\lambda) = x^{(0)\mu}(\lambda) +$

$x^{(1)\mu}(\lambda)$ is an affinely parametrized geodesic (not necessarily null) of ds^2 to first order provided that, for all λ_2, λ_1

$$x^{(1)\mu}(\lambda_2) = x^{(1)\mu}(\lambda_1) + (\lambda_2 - \lambda_1)\, k^{(1)\mu}(\lambda_1) + \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda)\, f^{(1)\mu}(\lambda)\, d\lambda \qquad (3)$$

where $k^{(1)\mu}(\lambda) = dx^{(1)\mu}(\lambda)/d\lambda$ and the integration is performed over $x^{(0)\mu}(\lambda)$.

The condition $k^{(0)0} = 1$ on the affine parametrization of the background geodesic in the above theorem is imposed simply so that we can use the propagators of Pyne and Birkinshaw (1995), who imposed that condition for ease of calculation. It is not hard to compute the necessary propagators for $k^{(0)\mu}(\tau)$ with τ any affine parameter. This is not really needed, however, since the geodesic, $k^{\mu}(\lambda)$, constructed by the theorem above may be reparametrized directly. For this reason we will sometimes refer to $k^{\mu}(\lambda)$ as a wavevector though this term is usually reserved for the tangent to a null geodesic parametrized so that $u_{\mu}k^{\mu}$ is the photon frequency observed by an observer with four-velocity u^{μ}.

Suppose now that we solve (3) along some given geodesic of $ds^{(0)2}$, $x^{(0)\mu}(\lambda)$ subject to $x^{(1)\mu}(\lambda_2) = 0$ and $x^{(1)i}(\lambda_1) = 0$. We can not simply demand that the separation, $x^{(1)\mu}(\lambda)$, vanish at both λ_1 and λ_2 if we want the constructed geodesic to be null since $x^{(0)\mu}(\lambda_1)$ and $x^{(0)\mu}(\lambda_2)$ are null separated in $ds^{(0)2}$ but not necessarily in ds^2. Because the spatial and timelike components of (3) decouple, however, we can use the above boundary conditions to obtain $k^{(1)i}(\lambda_1)$. We can then solve for $x^{(1)0}(\lambda_1)$ by demanding that our constructed geodesic be null.

The condition that our constructed geodesic be null in ds^2 can be written

$$k^{(1)0} = \frac{1}{2} k^{(0)\mu} h_{\mu\nu} k^{(0)\nu} + k^{(0)i} \eta_{ij} k^{(1)j} \qquad (4)$$

where the equation holds along $x^{(0)\mu}(\lambda)$. Taking our boundary conditions into account, this allows us to write the timelike component of (3) as

$$
\begin{aligned}
x^{(1)0}(\lambda_1) = -\frac{1}{2}(\lambda_2-\lambda_1)\, k^{(0)\mu} h_{\mu\nu} k^{(0)\nu} - (\lambda_2-\lambda_1)\, k^{(0)i}\eta_{ij}k^{(1)j} \\
- \int_{\lambda_1}^{\lambda_2} (\lambda_2-\lambda)\, f^{(1)0}(\lambda)\, d\lambda,
\end{aligned}
\tag{5}
$$

the inner products being evaluated at $x^{(0)\mu}(\lambda_1)$. The spatial components of (3) yield

$$
(\lambda_2-\lambda_1)\, k^{(1)i}(\lambda_1) = -\int_{\lambda_1}^{\lambda_2} (\lambda_2-\lambda)\, f^{(1)i}(\lambda)\, d\lambda \tag{6}
$$

which may be combined with (5) to produce

$$
x^{(1)0}(\lambda_1) = -\frac{1}{2}(\lambda_2-\lambda_1)\, k^{(0)\mu} h_{\mu\nu} k^{(0)\nu} + \int_{\lambda_1}^{\lambda_2} (\lambda_2-\lambda)\, k^{(0)}_{\mu} f^{(1)\mu}(\lambda)\, d\lambda \tag{7}
$$

Equation (7) could also have been obtained immediately from (3) and (4) after taking the inner product of (3) with $k^{(0)\mu}$.

At this point another representation of the perturbation vector, $f^{(1)\mu}$, is very useful. Letting a semicolon denote covariant differentiation using the Christoffel connection of $ds^{(0)2}$, we have

$$
f^{(1)\mu} = \frac{1}{2} h_{\alpha\beta}{}^{;\mu} k^{(0)\alpha} k^{(0)\beta} - h^{\mu}{}_{\alpha;\beta} k^{(0)\alpha} k^{(0)\beta}. \tag{8}
$$

Since $k^{(0)\mu}$ is geodesic, this gives

$$k^{(0)}_{\mu} f^{(1)\mu} = -\frac{1}{2}\frac{d}{d\lambda}\left(k^{(0)\mu} h_{\mu\nu} k^{(0)\nu}\right). \tag{9}$$

This may be substituted into (7) allowing an integration by parts to produce

$$x^{(1)0}(\lambda_1) = -\frac{1}{2}\int_{\lambda_1}^{\lambda_2}\left(k^{(0)\mu} h_{\mu\nu} k^{(0)\nu}\right) d\lambda \tag{10}$$

We have thus found the following

Corollary: Given two points, $q \equiv x^{(0)\mu}(\lambda_2)$ and $w \equiv x^{(0)\mu}(\lambda_1)$, connected by a null geodesic of $ds^{(0)2}$, $x^{(0)\mu}(\lambda)$, with $k^{(0)0} = 1$, then to first order the points $w' \equiv x^{(0)\mu}(\lambda_1) + \left(x^{(1)0}(\lambda_1), 0^i\right)$ and q are null separated in ds^2 along some geodesic x^μ provided $x^{(1)0}(\lambda_1)$ obeys (10). Further, the tangent vector to x^μ at w' is set by (4) and (6) above.

In Section 3 we will show how this corollary may be used to analyze the effects of an arbitrary metric perturbation on VLBI observations in the metric (1).

We hasten to point out that the theorem, and so corollary, above have no more content than a direct integration of the linearized geodesic equations of our spacetime. We have presented this information in this manner for two reasons. First, as we will demonstrate in the next section, the above corollary is specifically adapted to an analysis of VLBI experiments in the spacetimes we are considering. Second, the presentation above is organized so as to facilitate the generalization to curved backgrounds described in section 5 below. The reader will note that for scalar perturbations in the longitudinal gauge (see e.g. Mukhanov, Feldman, and Brandenberger 1992), the RHS of (10) is the Shapiro delay evaluated along the background path. We will return to the relationship between (10) and lens systems in section 6.

It remains for us to discuss the consistency criteria for the manipulations leading to our corollary. We defer an examination of this topic till section 6, contenting ourselves here with an informal remark. Imagine a geodesic of $ds^{(0)2}$, $\tilde{x}^{(0)\mu}(\rho)$ which intersects w' at affine parameter value λ_1 with tangent vector $\tilde{k}^{(0)\mu}_{w'} = k^{(0)\mu}_{w} + k^{(1)\mu}_{w}$, where a subscript w (w') denotes evaluation at w (w') and $k^{(1)\mu}_{w}$ is set by (4) and (6). We can use this geodesic in (3) to construct a null geodesic in ds^2, $\tilde{x}^{\mu}(\rho)$, obeying $\tilde{x}^{\mu}(\lambda_1) = w'$ and $\tilde{k}^{\mu}_{w'} = \tilde{k}^{(0)\mu}_{w'}$. At a minimum, then, we would expect (10) to be a consistent solution of the fixed endpoint problem only if $\tilde{x}^{\mu}(\lambda_2) = q$. Roughly speaking, we would expect this to hold if the gravitational effects of the perturbation are similar on $x^{(0)\mu}$ and $\tilde{x}^{(0)\mu}$ for equal affine parameter values, that is if x and $\tilde{x}$ pass through "sufficiently similar" metrics.

3. Gravitational Radiation and VLBI

In order to use (10) to investigate the effects of a gravitational wave background on a VLBI experiment we first need the appropriate form for the metric perturbation. The plane, monochromatic wave solution in the synchronous gauge to the perturbed Einstein's equations associated with (1) can be written as the real part of

$$\begin{aligned} h_{00} &= 0 \\ h_{0i} &= 0 \\ h_{ij} &= \frac{a_o}{a}\left[h_+(\vec{p})\left(Re_+R^T\right)_{ij} + h_\times(\vec{p})\left(Re_\times R^T\right)_{ij}\right] e^{i(\vec{p}\cdot\vec{x}-p\eta)} \\ &= \frac{a_o}{a}H_{ij}e^{i(\vec{p}\cdot\vec{x}-p\eta)} \end{aligned} \tag{11}$$

(Hawking, 1966), where a_0 is some fiducial value of a, h_+ and $h_\times$ are complex valued functions containing the amplitude and phase information, e_+ and $e_\times$ are polarization matrices which we represent by

$$\begin{aligned} e_+ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ e_\times &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}, \tag{12}$$

R is a rotation matrix, R^T its transpose, which we represent by

$$R(\theta,\phi)=\begin{pmatrix} \sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ -\cos\phi & \cos\theta\sin\phi & \sin\theta\sin\phi \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{13}$$

and the modevector,

$$\begin{aligned} p^i &= p\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\ &= R\begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} \end{aligned}. \tag{14}$$

We can think of the angles, θ and ϕ, as functions of $\vec{p}$ through (14). This solution is valid in what is known as the adiabatic regime, which demands that, at the times of interest, the physical reduced wavelength of the wave is much smaller than the Hubble distance. Mathematically this is simply $1 \gg a'/p$ where $' \equiv d/dt$, t the comoving time co-ordinate related to η by $d/dt = a^{-1}d/d\eta$. The comoving reduced wavelength, $1/p$, is related to the physical wavelength on a spatial hypersurface of constant conformal time η, λ_{phys}, by $\lambda_{\text{phys}} = 2\pi a(\eta)/p$. The condition that a wave be in the adiabatic regime at (unperturbed) redshift z is thus $\lambda_{\text{phys}} \ll 2\pi(1+z)^{-3/2}H_o^{-1}$, with H_o the Hubble constant at $z=0$. The general tensor perturbation in the adiabatic regime, $h_{\mu\nu}(\eta,\vec{x})$, is a superposition of the plane, monochromatic waves for different modevectors.

We now show how to use the corollary of section 2 to analyze a VLBI experiment in a metric perturbed Einstein-de Sitter spacetime with perturbation given by (11). The basic idea is to construct the future null cone of a point of emission by determining the time at which the cone intersects any given line of constant spatial position. This will allow us easily to determine the difference in time of reception of a signal at two points a known spatial distance apart, that is, at two ends of an interferometer. The geometry of this section is shown in Figure 1.

Suppose an antenna, which we label antenna A, receives at the point $w'_A \equiv (\eta'_A, \vec{r}_A)$ photons from a source at $q \equiv (\eta_o - L, L\hat{e}_s)$. We take $\hat{e}_s$ to be normalized to unity in $ds^{(0)2}$. Denote the translation of w'_A through time by $\tau_A \equiv (\eta, \vec{r}_A)$. The forward null cone in $ds^{(0)2}$ of q intersects τ_A at the point $w_A = (\eta_o + \Delta_A, \vec{r}_A)$ where

$$\Delta_A = |L\hat{e}_s - \vec{r}_A| - L. \tag{15}$$

In (15) and below, the $ds^{(0)2}$ inner product is written using vertical bars. We will also use a raised dot for this inner product. We note that Δ_A has a very simple interpretation: two antennae, one at the spatial origin and one at spatial coordinates given by $\vec{r}_A$, observing a source at q in an unperturbed Einstein-de Sitter spacetime would measure a time delay of Δ_A between the reception of a signal at the origin at time η_o and the reception of the same signal by the antenna at $\vec{r}_A$. The connecting null geodesic is given by $x_A^{(0)\mu}(\lambda) = (\lambda, L\hat{e}_s - (\lambda - \eta_o + L)\,\hat{e}_A)$ with

$$\hat{e}_A = \frac{L\hat{e}_s - \vec{r}_A}{|L\hat{e}_s - \vec{r}_A|} \tag{16}$$

which is seen to be properly normalized. The corollary of section 2 now demands

$$\eta'_A = \eta_o + \Delta_A - \frac{1}{2}\int_{\eta_o+\Delta_A}^{\eta_o-L} \left(k^{(0)\mu} h_{\mu\nu} k^{(0)\nu}\right)\, d\lambda \tag{17}$$

the integral being taken over $x_A^{(0)\mu}(\lambda)$.

It will be convenient for us to use not (17), but the equivalent expression in terms of the comoving time. Provided that the change in the scale factor over Δ_A and $\eta_A^{(1)}$ may be neglected

$$\begin{aligned} t'_A &= t_o + a_o\Delta_A + t_A^{(1)} \\ &= t_o + a_o\Delta_A - \frac{a_o}{2}\int_{\eta_o+\Delta_A}^{\eta_o-L} \left(k^{(0)\mu} h_{\mu\nu} k^{(0)\nu}\right)\, d\lambda, \end{aligned} \tag{18}$$

the integration occuring over $x_A^{(0)\mu}(\lambda)$.

We now consider a second antenna, antenna B, which receives photons at $w'_B \equiv (\eta_B, \vec{r}_B)$ from the same point of emission, q. This system obeys equations (15) and (18) when all of the subscript A's are replaced by subscript B's. The measured (comoving) time delay for the two antenna system is given by

$$\begin{aligned} T_d &= t'_A - t'_B \\ &= a_o \left(\Delta_A - \Delta_B\right) + t_A^{(1)} - t_B^{(1)} \end{aligned} \tag{19}$$

with $t_A^{(1)}$ given by the term in (18) linear in $h_{\mu\nu}$ and $t_B^{(1)}$ given by the analogous expression for antenna B.

To see that (19) contains the usual expression for the time delay of a VLBI system, consider the term, $T_d^{(0)}$, in (19) which is independent of the perturbation. Put $\vec{r}_B = \vec{r}_A + \vec{b} = \vec{r} + \vec{b}$. We take the distance to the source to be much larger than the proper lengths of either $\vec{r}$ or $\vec{b}$, and we work to first order in the implied small quantities, e.g $\vec{b}/L$. This is the mathematical expression of the locally plane wave approximation whereby we neglect the curvature of the wavefronts at the observer. With this approximation we have $\hat{e}_s \approx \hat{e}_A$ and

$$a_o \left(\Delta_A - \Delta_B\right) \approx a_o \vec{b} \cdot \hat{e}_s \tag{20}$$

Continuing, we note that the scale factor $a_o \approx a\left(t'_A\right) \equiv a(t)$ where we have written the time of reception at A, t'_A, simply as t. Provided antenna B does not move significantly over T_d we can write $\vec{b}$ as $\vec{b}(t)$, the (comoving) co-ordinate vector reaching from antenna A to antenna B at time t. In addition, we can give meaning to $\hat{e}_s(t)$ as follows: the spatial co-ordinates of the source at the event of emission of the photons which arrive at antenna A at time t is given by $L(t)\hat{e}_s(t)$, with $\hat{e}_s(t)$ normalized to unity in $ds^{(0)2}$. The first term in (19) can then be written $a\vec{b} \cdot \hat{e}_s$, with each quantity evaluated at t.

The tensor which projects the wavevector of the arriving photons $\bar{k}^\mu$ into the rest space of the observing antennae is written $\delta^\mu_\nu + u^\mu u_\nu$, with u^μ the antennae four-velocity. Taking our antennae comoving (for simplicity, a Lorentz boost will easily yield the general case from the specific) the properly normalized (in $d\bar{s}^2$)

four-velocity is given by $u^\mu = (1/a, \vec{0})$. This writes the unit (in $d\bar{s}^2$) vector in the rest space of the observer which points toward the source, s^μ, as

$$s^\mu = -\frac{\bar{k}^\mu}{\left|u\bar{\cdot}\bar{k}\right|} - u^\mu \frac{\left(u\bar{\cdot}\bar{k}\right)}{\left|u\bar{\cdot}\bar{k}\right|}. \tag{21}$$

It is easy to check that s^μ has no timelike component. In (21) we have used $\bar{\cdot}$ to denote the $\bar{g}_{\mu\nu}$ inner product. Since $\bar{k}^\mu = a^{-2}k^\mu = a^{-2}k^{(0)\mu} + a^{-2}k^{(1)\mu}$ we have

$$\begin{aligned} as^i &= \hat{e}^i - k^{(1)i} - k^{(1)0}\hat{e}^i \\ &= \hat{e}^i - k^{(1)i} - \frac{1}{2}\hat{e}^m h_{mn}\hat{e}^n\hat{e}^i + \hat{e}_m k^{(1)m}\hat{e}^i \end{aligned} \tag{22}$$

where we have used (4) and we have written $\hat{e}_A$ simply as $\hat{e}$. At zeroth order this allows us to write (19) as

$$\begin{aligned} T_d^{(0)} &= a_o^2 b^i \eta_{ij} s^{(0)j} \\ &= \left(b^i \bar{g}_{ij} s^j\right)^{(0)}. \end{aligned} \tag{23}$$

This formula, scalar under spatial transformations in the antenna rest space and whose quantities are simply the restriction to this space of tensor quantities (which by definition have well defined properties under Lorentz transformation), is exactly the standard formula of VLBI (Thompson, Moran, and Swenson 1986).

Under the locally plane wave approximation, $\left(b^i \bar{g}_{ij} s^j\right)$ is the spatial distance in the observer's rest frame that the incoming wavefront must still travel to antenna B at the instant it has hit antenna A. It is, thus, the time delay. In this approximation, then, we can obtain the time delay either by determining the perturbed direction vector to the emitter, s^i, and expanding its inner product with the baseline vector, b^i, or we can work directly from (19), imposing the mathematical constraints which lead to locally plane waves at the observer. We now show that these methods agree by calculating the explicit formula for the time delay using both methods

We have already seen that the zeroth order expressions agree. Unfortunately, the proof to first order is not so simple. We start by setting $a_0 = 1$ in order to simplify our notation. This means that at w_A there is no distinction between $d\bar{s}^2$ and ds^2. Since we will only be concerned with quantities at w_A in our formulae, we will, for the rest of this section, no longer put a bar over the physical metric. Nevertheless, its presence is implied, that is, e.g. $h_{\mu\nu}$ may be understood as $\bar{h}_{\mu\nu}$, the two being equal at the point we need them.

We first calculate the first order term in the time delay from the formula

$$T_d^{(1)} = \left(b^i g_{ij} s^j\right)^{(1)} \tag{24}$$

From (22) we see that we need the perturbation to the spatial components of the wavevector at antenna A, $k_A^{(1)i}$. This quantity is solved for by (6), above, using the background path $x_A^{(0)\mu}$, with the result

$$\begin{aligned} k_A^{(1)i} = & \left(\frac{i}{2}\hat{e}_A \cdot H\hat{e}_A\vec{p} - ip\left(1+\hat{p}\cdot\hat{e}_A\right)H\hat{e}_A\right)\frac{\eta_o^2}{L+\Delta_A}e^{i\vec{p}\cdot\vec{r}} \\ & \times e^{i\vec{p}\cdot\hat{e}_A(\eta_o+\Delta_A)}J \end{aligned} \tag{25}$$

with

$$J = (\eta_o - L)\int_{\eta_o+\Delta_A}^{\eta_o-L} \lambda^{-2} e^{-ip(1+\hat{p}\cdot\hat{e}_A)\lambda}\,d\lambda - \int_{\eta_o+\Delta_A}^{\eta_o-L} \lambda^{-1} e^{-ip(1+\hat{p}\cdot\hat{e}_A)\lambda}\,d\lambda \tag{26}$$

The integrals may be performed explicitly, resulting in combinations of exponential and exponential-integral functions of arguments $-ip\left(1+\hat{p}\cdot\hat{e}_A\right)(\eta_0 - L)$ and $-ip\left(1+\hat{p}\cdot\hat{e}_A\right)(\eta_0 + \Delta_A)$. The result, while correct, is not particularly easy to use and we do not write it down. Instead, we note that adiabaticity demands

$p(\eta_0 - L) \gg 1$, which allows us to use the large argument expansion for the exponential-integral functions (e.g. Gradshteyn and Ryzhik 1994, equation 8.215) for all values of $\hat{e}_A$ not too close to $-\hat{p}$. Next we suppose that our source is many gravitational wave (reduced) wavelengths away, $pL \gg 1$, and we agree to work only to leading order in $(pL)^{-1}$. The solution, for the appropriate range of $\hat{e}_A$, then becomes

$$
\begin{aligned}
s^i = \hat{e}^i_A + & \frac{\eta_0^2}{(\eta_0 + \Delta_A)^2} e^{i\vec{p}\cdot\vec{r}_A} e^{-ip(\eta_0+\Delta_A)} \\
& \times \frac{\hat{e}_A \cdot H \hat{e}_A}{2\,(1 + \hat{p}\cdot\hat{e}_A)} (\hat{e}_A + \hat{p}) \ .
\end{aligned}
\tag{27}
$$

Because this expression shares with the exact result, (25) and (26) above, the property of vanishing at $\hat{e}_A = -\hat{p}$ we will use it for all values of $\hat{e}_A$. Rigorously, we should examine its approach to this zero and compare it with that of the exact expression, but we will consider ourselves justified by the end result. Noting that, to first order,

$$
\begin{aligned}
h_{ij}(w'_A) &= h_{ij}(w_A) \\
&= \frac{\eta_0^2}{(\eta_0 + \Delta_A)^2} e^{i\vec{p}\cdot\vec{r}_A} e^{-ip(\eta_0+\Delta_A)} H_{ij} \ ,
\end{aligned}
\tag{28}
$$

equation (27) writes for the time delay

$$
\begin{aligned}
T_d &= \left(b^i g_{ij} s^j\right) \\
&= \vec{b}\cdot\hat{e}_A + \frac{1}{2}\hat{e}^i_A h_{ij} \hat{e}^j_A \frac{\vec{b}\cdot\hat{e}_A + \vec{b}\cdot\hat{p}}{(1 + \hat{p}\cdot\hat{e}_A)}
\end{aligned}
\tag{29}
$$

We note that this formula is a well-defined scalar in rest space of our observer. We also point out that $\hat{p} = \hat{p}_{\rm phys}$ and that our hatted vectors are of unit norm in the physical background metric at the observer since, briefly returning to our old bar notation, $\bar{g}^{(0)}_{\mu\nu} = \eta_{\mu\nu}$. Defining the dimensionless vector $\vec{\zeta} = \hat{e}_A + \hat{p}$, with $\hat{\zeta} \equiv \vec{\zeta}/\sqrt{\zeta^i \eta_{ij} \zeta^j}$, we have

$$T_d = \vec{b} \cdot \hat{e}_A + \vec{b} \cdot \vec{\zeta} \left(\hat{\zeta}^i h_{ij} \hat{\zeta}^j \right). \tag{30}$$

In the following section we will show how this formula may be used to obtain the pattern of proper motions on the sky measured by an interferometer observing a sample of distant sources through a background of gravitational radiation.

For completeness, we now show how our solution, (30), is obtained from (18) and (19), above. These write the first order contribution to the time delay as

$$\begin{aligned} T_d^{(1)} &= t_A^{(1)} - t_B^{(1)} \\ &= -\frac{1}{2} \int_{\eta_\circ + \Delta_A}^{\eta_\circ - L} k_A^{(0)\mu} h_{\mu\nu} \left(x_A^{(0)} \right) k_A^{(0)\nu} \, d\lambda + \frac{1}{2} \int_{\eta_\circ + \Delta_B}^{\eta_\circ - L} k_B^{(0)\mu} h_{\mu\nu} \left(x_B^{(0)} \right) k_B^{(0)\nu} \, d\lambda. \end{aligned} \tag{31}$$

In this formula, the path on which $h_{\mu\nu}$ is to be evaluated has been shown explicitly.

The easiest way to proceed is to expand the integrand in the integral for $t_B^{(1)}$ about the value of the integrand in the integral for $t_A^{(1)}$ for fixed λ. If we neglect terms of quadratic order in $\vec{b}/L$,

$$\begin{aligned} k_B^{(0)\mu} h_{\mu\nu} \left(x_B^{(0)} \right) k_B^{(0)\nu} = k_A^{(0)\mu} h_{\mu\nu} \left(x_A^{(0)} \right) k_A^{(0)\nu} + 2 \delta k^{(0)\mu} h_{\mu\nu} \left(x_A^{(0)} \right) k_A^{(0)\nu} \\ + k_A^{(0)\mu} h_{\mu\nu,\alpha} \left(x_A^{(0)} \right) \delta x^{(0)\alpha} k_A^{(0)\nu} \end{aligned} \tag{32}$$

with

$$\begin{aligned} \delta k^{(0)\mu} &= k_B^{(0)\mu} - k_A^{(0)\mu} \\ &= \frac{1}{L} \begin{pmatrix} 0 \\ \vec{b} - (\vec{b} \cdot \hat{e}_s) \hat{e}_s \end{pmatrix} \end{aligned} \tag{33}$$

and $\delta x^{(0)\mu}(\lambda) = (\lambda + L - \eta_0)\, \delta k^{(0)\mu}$. Using (32) in (31) yields

$$
\begin{aligned}
T_d^{(1)} &= \frac{1}{2} \int_{\eta_0+\Delta_A}^{\eta_0-L} 2\delta k^{(0)\mu} h_{\mu\nu} \left(x_A^{(0)}\right) k_A^{(0)\nu} + k_A^{(0)\mu} h_{\mu\nu,\alpha} \left(x_A^{(0)}\right) \delta x^{(0)\alpha} k_A^{(0)\nu}\, d\lambda \\
&\quad + \frac{1}{2} k_A^{(0)\mu} h_{\mu\nu} \left(p_A\right) k_A^{(0)\nu} \\
&= \frac{1}{2} k_A^{(0)\mu} h_{\mu\nu} \left(p_A\right) k_A^{(0)\nu} \\
&\quad - \hat{e}_A \cdot H \left(\vec{b} - (\vec{b} \cdot \hat{e}_s)\hat{e}_s\right) \frac{\eta_0^2}{L} e^{i\vec{p}\cdot\vec{r}_A} e^{-i\vec{p}\cdot\hat{e}_A(\eta_0+\Delta_A)} I \\
&\quad - \frac{i}{2} \hat{e}_A \cdot H \hat{e}_A \vec{p} \cdot \left(\vec{b} - (\vec{b} \cdot \hat{e}_s)\hat{e}_s\right) \frac{\eta_0^2}{L} e^{i\vec{p}\cdot\vec{r}_A} e^{-i\vec{p}\cdot\hat{e}_A(\eta_0+\Delta_A)} J
\end{aligned}
\tag{34}
$$

where J is given by (26) and

$$
I = \int_{\eta_o+\Delta_A}^{\eta_o-L} \lambda^{-2} e^{-ip(1+\hat{p}\cdot\hat{e}_A)\lambda}\, d\lambda. \tag{35}
$$

Once again, the integrals we need to perform yield a combination of exponential and exponential-integral functions of arguments $-ip\,(1+\hat{p}\cdot\hat{e}_A)\,(\eta_0 - L)$ and $-ip\,(1+\hat{p}\cdot\hat{e}_A)\,(\eta_0+\Delta_A)$. We follow our earlier course and use the large argument expansion of the exponential-integral functions. The terms in the resultant expression contributed by the term written proportional to I in (34), are of order $1/pL$ times the terms contributed by the term written proportional to J in (34), and of order $1/pL$ times the first term in (34) (in fact, direct dimensional analysis of (34) argues for this conclusion). Again restricting ourselves to sources which are many gravitational wave reduced wavelengths away, we may neglect the terms down by pL. The remaining terms combine easily to give the first order part of (29), which is what we needed to show. That is, we have proven, through first order,

$$
t_A - t_B = b^i g_{ij} s^j \tag{36}
$$

with the LHS assembled from (15), (18), and (19), and the RHS constructed from (22).

Because this has been a long section we will collect the approximations used to gain our formula here. We envision an adiabatic background of gravitational radiation and two antennae, A and B, such that the co-ordinate separation vector, $\vec{b}$, reaching from antenna A to antenna B has a proper length much smaller than the (reduced) wavelengths of the waves, $p\vec{b} \ll 1$. We suppose the antennae to be nearly comoving over the timescale of the measured delay. We further suppose that the sources observed are many gravitational wave (reduced) wavelengths away, $pL \gg 1$. Provided that these inequalities hold, the measured time delay, T_d, is given by (29), with $\hat{e}_A$ related to the observed direction vector to the source, s^i, by (22) (using (25) as well). We note that all of our conditions may be relaxed if necessary, by using the more precise equations (19) and (34).

We can obtain a very rough *a posteriori* consistency check on the accuracy of our perturbative expansion in the following way. We approximate the separation between the solution geodesic, $x^\mu(\lambda)$, and $x^{(0)\mu}(\lambda)$ by $Lk^{(1)}(\lambda_1) \sim Lh$, where h is some characteristic element of $h_{\mu\nu}$. That is, the separation is roughly the distance to the source times the angle between the two geodesics at the observer. We must demand that the perturbations "felt" by these two geodesics are nearly identical, which means, since $h_{\mu\nu}$ varies over scales larger than $1/p$, $Lh \ll p^{-1}$. The idea is now to square this inequality and relate h^2p^2 to the fraction of the closure density contained in the gravitational radiation. An explicit model for the wave background is necessary for this step. For illustration, consider a stochastic background of gravitational radiation. Then $\Omega_{\mathrm{GW}} \sim p^2h^2/H_0^2$. Since

$$L = 2H_0^{-1}\left(1-(1+z)^{-1/2}\right) \tag{37}$$

we can write our constraint as

$$\Omega_{\mathrm{GW}} \ll \frac{1}{4}\left(1-(1+z)^{-1/2}\right)^{-2} \tag{38}$$

This is to be interpreted as follows. If we use the formulae above to infer, from sources at characteristic redshift z, the presence of a stochastic background of gravitational waves with an energy density of Ω_{GW}, then a crude estimate of consistency is furnished by the degree which (38) is found to hold. We point out that for $z \sim 1$ this is not a highly restrictive condition.

4. The Pattern of Proper Motions on the Sky

In this section we will show how equation (30) may be used to gain the pattern of proper motions inferred by an interferometrist observing distant sources through a background of gravitational radiation. Since we are working in the linear regime, without loss of generality we consider the effect of a single plus-polarized monochromatic wave. The effect of a general wave background may be found from the results of this section using superposition. We choose our coordinates such that the z-axis is aligned with the waves direction of propagation. The modevector, $\vec{p}$, is then given by $\vec{p} = p\hat{z}$ and the metric perturbation is written

$$\begin{aligned} h_{00} &= 0 \\ h_{0i} &= 0 \\ h_{ij} &= \frac{1}{a} h_+ e^{ip(z-\eta)} \left(e_+\right)_{ij} \end{aligned} . \tag{39}$$

As in the last section we have set $a_0 = 1$. Unlike there, however, in this section we return to our practice of using an overbar to distinguish between the perturbed Einstein-de Sitter spacetime and its conformal relative, shown in (1).

The approximations of the previous section guarantee the existence of a region of spacetime Σ such that Σ contains the events of reception, $a \approx 1$ in Σ, and $px^i \ll 1$ for any $x^\mu \in \Sigma$. This last condition simply states that the spatial extent of Σ on any constant conformal time hypersurface is much smaller than the wavelength of the gravitational wave. In Σ, to within our level of approximation, the coordinate transformation

$$
\begin{aligned}
t' &= t \\
x' &= \left(1 + \frac{1}{2} h_+ e^{ip(z-\eta)}\right) x \\
y' &= \left(1 - \frac{1}{2} h_+ e^{ip(z-\eta)}\right) y \\
z' &= z
\end{aligned}
\tag{40}
$$

brings the actual metric to Minkowski form; $\bar{g}_{\mu'\nu'} = \eta_{\mu'\nu'}$. We note that in (40) η is to be considered a function of t in the usual way.

For simplicity we again suppose at least one of the antennae is comoving, having four velocity $u^\mu = (1, 0, 0, 0)$. The surfaces of constant conformal time then serve as instantaneous rest three-spaces for an observer at this antenna. The co-ordinate transformation (40) may be considered a transformation of co-ordinates in these hypersurfaces. Since the spatial components of s^i, given by (21) above, define a vector in these hypersurfaces, the observed direction to a source in the primed coordinates is simply

$$
s^{i'} = \frac{\partial x^{i'}}{\partial x^i} s^i \ , \tag{41}
$$

and the time delay measured by an antenna pair with separation $b^{i'}$ is given by

$$
T_d = b^{i'} \eta_{i'j'} s^{i'} \ . \tag{42}
$$

Consider now the triad of vectors

$$
\begin{aligned}
b_{(1)} &= B\left(1 - \frac{1}{2}h_+ e^{-ip\eta}\right)\hat{x} \\
b_{(2)} &= B\left(1 + \frac{1}{2}h_+ e^{-ip\eta}\right)\hat{y} \\
b_{(3)} &= B\hat{z}
\end{aligned}
\tag{43}
$$

with $\hat{x}$, $\hat{y}$, and $\hat{z}$ the unprimed (spatial) co-ordinate basis vectors (so, for example, $\hat{y}^\mu = (0, 0, 1, 0)$). The vectors $b_{(i)}$ are orthogonal and have proper length B with respect to $\bar{g}_{\mu\nu}$. Their spatial components define a triad of three-vectors in the observers rest space, orthogonal and of proper length B with respect to the induced metric $\bar{g}_{ij}$. In terms of the primed (spatial) co-ordinate basis vectors, $\hat{x}'$, $\hat{y}'$, and $\hat{z}'$, the triad is written

$$
\begin{aligned}
b_{(1)} &= B\hat{x}' \\
b_{(2)} &= B\hat{y}' \\
b_{(3)} &= B\hat{z}'
\end{aligned}
\tag{44}
$$

We now suppose that the observer makes time delay measurements of a source with $\hat{e}_A = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}$ using each of the three-vectors of the triad as a baseline. Then (30), (39), and (43) tell us that $T_{d(i)}$, the time delay measured along baseline $\vec{b}_{(i)}$, is given by

$$
\begin{aligned}
T_{d(1)} &= \alpha B\left(1 - \frac{1}{2}h_+ e^{-ip\eta}\right)\left[1 + \frac{1}{2}h_+ e^{-ip\eta}\left(\frac{\alpha^2 - \beta^2}{1+\gamma}\right)\right] \\
T_{d(2)} &= \beta B\left(1 + \frac{1}{2}h_+ e^{-ip\eta}\right)\left[1 + \frac{1}{2}h_+ e^{-ip\eta}\left(\frac{\alpha^2 - \beta^2}{1+\gamma}\right)\right]. \\
T_{d(3)} &= B\left[\gamma + \frac{1}{2}h_+ e^{-ip\eta}\left(\alpha^2 - \beta^2\right)\right]
\end{aligned}
\tag{45}
$$

Since the time delay is a scalar under co-ordinate transformations in the observer's rest space we can also compute the delays in the primed co-ordinates,

$$T_{d(i)} = b^{m'}_{(i)} \eta_{m'n'} s^{n'}. \tag{46}$$

Using (44) in (46) and equating the resulting expression with the right-hand side of (45) allows us to read off the components of the normalized direction vector to the source in the primed co-ordinates,

$$\begin{aligned} s^{x'} &= \alpha \left(1 - \frac{1}{2} h_+ e^{-ip\eta}\right) \left[1 + \frac{1}{2} h_+ e^{-ip\eta} \left(\frac{\alpha^2 - \beta^2}{1+\gamma}\right)\right] \\ s^{y'} &= \beta \left(1 + \frac{1}{2} h_+ e^{-ip\eta}\right) \left[1 + \frac{1}{2} h_+ e^{-ip\eta} \left(\frac{\alpha^2 - \beta^2}{1+\gamma}\right)\right] . \\ s^{z'} &= \left[\gamma + \frac{1}{2} h_+ e^{-ip\eta} \left(\alpha^2 - \beta^2\right)\right] \end{aligned} \tag{47}$$

The proper motion inferred by the observer is given by $\mu^i = P^{i'}{}_{\beta'} u^{\alpha'} s^{\beta'}{}_{|\alpha'}$ where $P^{\alpha'}{}_{\beta'} = \delta^{\alpha'}_{\beta'} + u^{\alpha'} u_{\beta'}$ is the projector into the observer's rest space and a slash denotes the covariant derivative of the perturbed metric, $d\bar{s}^2$. To first order this results in the simple $\mu^{i'} = s^{i'}{}_{,t'}$. A comoving source will have α, β, and γ constant. Then

$$\begin{aligned} \vec{\mu} = & \frac{ip\alpha}{2} h_+ e^{-ip\eta} \left(1 - \frac{\alpha^2 - \beta^2}{1+\gamma}\right) \hat{x}' \\ & - \frac{ip\beta}{2} h_+ e^{-ip\eta} \left(1 + \frac{\alpha^2 - \beta^2}{1+\gamma}\right) \hat{y}' . \\ & - \frac{ip}{2} h_+ e^{-ip\eta} \left(\alpha^2 - \beta^2\right) \hat{z}' \end{aligned} \tag{48}$$

Allowing the direction cosines to depend on time enables the effect of source peculiar motion to be calculated. To the level of accuracy considered here the effect of source motion is to superpose the standard FRW proper motion results on the gravitational wave pattern given by (48) (see e.g. Weinberg 1972, Chapter 15).

The expression, (48), for the proper motion contains the direction cosines α, β, and γ which are the components of the unobservable vector $\hat{e}_A$ in the unprimed co-ordinates. These differ from the direction cosines in the primed co-ordinates, α', β', and γ' where $\hat{e}_A = \alpha'\hat{x}' + \beta'\hat{y}' + \gamma'\hat{z}'$, by terms of first order. Further, as can be seen from (47), α, β, and γ describe the components of the observed direction vector to the quasar, s, to zeroth order in the primed (and unprimed) co-ordinates. Since $\vec{\mu}$ is entirely first order we may reinterpret α, β and γ in (48) as the direction cosines to the source in the primed co-ordinates, the difference between these quantities being first order and so resulting only in an ignorable second order correction to $\vec{\mu}$. In terms of the spherical polar co-ordinates of the source in the primed frame, defined by $s = \sin\theta' \cos\phi'\hat{x}' + \sin\theta' \sin\phi'\hat{y}' + \cos\theta'\hat{z}'$, we have

$$\begin{aligned} \alpha &\approx \sin\theta' \cos\phi' \\ \beta &\approx \sin\theta' \sin\phi' \\ \gamma &\approx \cos\theta' \end{aligned} \tag{49}$$

the approximate sign denoting equality to zeroth order. Substitution of (49) into (48) and resolution of the resultant expression into components along the spherical polar basis vectors given by

$$\begin{aligned} \hat{\theta}' &= \cos\theta' \cos\phi'\hat{x}' + \cos\theta' \sin\phi'\hat{y}' - \sin\theta'\hat{z}' \\ \hat{\phi}' &= -\sin\phi'\hat{x}' + \cos\phi'\hat{y}' \end{aligned} \tag{50}$$

produces the result

$$\vec{\mu} = \frac{ip}{2} h_+ e^{-ip\eta} \sin\theta' \left(\cos 2\phi'\,\hat{\theta}' - \sin 2\phi'\,\hat{\phi}'\right) \tag{51}$$

expressing the inferred proper motion in terms of the wave parameters, p and h_+, and the angular co-ordinates of the source in a gaussian normal frame (to the necessary order of approximation) at the observer.

5. Generalization to Curved Backgrounds

In this section, we will consider the general context for the calculations of the preceeding sections. By using the Jacobi equation of the background spacetime, we will be able to obtain a physical description of the manipulations involved. We begin by recalling that the perturbative geodesic expansion (PGE) constructs an affinely parametrized geodesic of a general perturbed metric, $g^{(0)}_{\mu\nu} + h_{\mu\nu}$, from an affinely parametrized geodesic, $x^{(0)\mu}(\lambda)$, of the background metric, $g^{(0)}_{\mu\nu}$, by writing the sought after geodesic as $x^{\mu}(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)$ and solving for $x^{(1)\mu}(\lambda)$, the separation (Pyne and Birkinshaw 1993). Let the Jacobi propagator along $x^{(0)\mu}(\lambda)$ be written in terms of its 4×4 subblocks as

$$U(\lambda_1, \lambda_2) = \begin{pmatrix} A(\lambda_1, \lambda_2) & B(\lambda_1, \lambda_2) \\ C(\lambda_1, \lambda_2) & D(\lambda_1, \lambda_2) \end{pmatrix} . \tag{52}$$

Then the equation governing the separation is written

$$\begin{aligned} P(\lambda_1, \lambda_2)\, x^{(1)}(\lambda_2) = {} & A(\lambda_1, \lambda_2)\, x^{(1)}(\lambda_1) + B(\lambda_1, \lambda_2)\, k^{(1)}(\lambda_1) \\ & + B(\lambda_1, \lambda_2) \left[\frac{d}{d\lambda} P(\lambda_1, \lambda) \right]_{\lambda=\lambda_1} x^{(1)}(\lambda_1) \\ & + \int_{\lambda_1}^{\lambda_2} B(\lambda_2, \lambda)\, P(\lambda_1, \lambda)\, f^{(1)}(\lambda)\, d\lambda \end{aligned} \tag{53}$$

where $k^{(1)} = dx^{(1)}/d\lambda$, P is the parallel propagator along $x^{(0)\mu}(\lambda)$, and $f^{(1)}$ is the perturbation vector given by (8), above. In (53), we have employed a 4×4 matrix notation so that, for example,

$$P(\lambda_1, \lambda)\, f^{(1)}(\lambda) \equiv P(\lambda_1, \lambda)^{\mu}{}_{\nu} f^{(1)\nu}(\lambda) \; . \tag{54}$$

Specific forms for the Jacobi and parallel propagators of the curved FRW spacetimes can be found in Pyne and Birkinshaw (1995).

Equation (53) is the generalization of equation (3) of section 2. The proper generalization of equation (4) of section 2, the condition that $k^{\mu}(\lambda)$ be null to first order in the perturbed metric, is written

$$h_{\mu\nu}k^{(0)\mu}k^{(0)\nu} + 2g^{(0)}_{\mu\nu}k^{(0)\mu}k^{(1)\nu} + 2g^{(0)}_{\mu\nu,\rho}x^{(1)\rho}k^{(0)\mu}k^{(0)\nu} = 0, \tag{55}$$

the constraint holding at each point of $x^{(0)\mu}(\lambda)$ provided it holds at any given point. In principle, equations (53) and (55) allow us to carry out analysis of the geodesic problem subject to fixed-endpoint (or mixed) boundary conditions in arbitrary metric perturbed spacetimes. Such boundary conditions are more applicable to certain astrophysical systems, such as multiple image lensing or VLBI, than the more common initial-value boundary data which is useful, for instance, in studies of the Sachs-Wolfe effect (Sachs and Wolfe 1967).

The primary use we will make of equations (53) and (55) in this paper, however, is to understand the manipulations of sections 2 and 3. The physical picture is made clear by recognizing that (53) is exactly the Jacobi equation of the background spacetime subject to a forcing perturbation $f^{(1)}$. Consider the fixed endpoint solution between q and w'_A which made use of the unperturbed geodesic $x^{(0)\mu}(\lambda)$ between q and w_A. We will simply state the conclusion; the reader can carry out the steps explicitly and compare them with the development in sections 2 and 3 to confirm the conclusion that we offer here. The solution is constructed in the following way. First, solve, using (53), for the spatial separation, $\delta x\,(\lambda_2)$, on the constant conformal time hypersurface containing the point of emission, q, attained by a perturbed null geodesic which intersects w_A with wavevector coincident with $k^{(0)\mu}\,(\lambda_1)$. We can think of this as tracing the photon into the past and determining its intersection with the constant conformal time hypersurface containing q. The proper pertubation to the spatial components of the wavevector at w_A, $k^{(1)i}\,(\lambda_1)$, is then taken to be that perturbation at w_A which, when considered as an impulsive perturbation in the background Jacobi equation, produces a deviation vector on the constant conformal time hypersurface containing q equal to $\delta x\,(\lambda_2)$. Speaking in informal language borrowed from gravitational lens theory, we solve a standard initial-data problem to gain a spatial separation in the source plane, then we convert that to an angle at the observer using the angular diameter distance of the background.

From the spatial components, $k^{(1)i}\,(\lambda_1)$, we determine the timelike component of the wavevector perturbation, $k^{(1)0}\,(\lambda_1)$, by imposition of the null constraint (55). Imagine we now constructed a null geodesic of $g^{(0)}_{\mu\nu}$ which intersected w_A

with wavevector $k^{(0)\mu}(\lambda_1) + k^{(1)\mu}(\lambda_1)$ and used this geodesic to obtain a perturbed geodesic which intersects w_A with coincident wavevector. Modulo questions of consistency, which we will discuss below, the resultant perturbed geodesic will intersect a time translate of q but not, generally, q itself. That is, it will "hit" the proper spatial co-ordinates but will "miss" the point q by some offset in time. We correct for this by "moving" the point w_A in time to another point w'_A. The necessary time translation is $x^{(1)0}(\lambda_1)$.

It is clear that great simplifications have resulted from the foliation of our background spacetime by spatial hypersurfaces. While the FRW backgrounds possess such a foliation a general background does not, and we have not considered fixed-endpoint problems in such spacetimes. In the next section we return to general questions of consistency, using as a theoretical laboratory the classic astrophysical example of a fixed-endpoint solution, the gravitational lens.

6. A Simple Gravitational Lens and Consistency

We work in a Minkowski space background. We consider an observer at the spatial origin of co-ordinates, a lens, of mass m, located at spatial co-ordinates $L\hat{x}$, and an emitter which emits a burst of photons at spacetime event $q = (-2L, 2L, 0, 0)$. Figure 2 illustrates our geometry. The question we want to answer is where on the sky does the observer see the emitter? The answer, of course, is that the observer sees a circular ring around the lens, with the angle between lens and ring given by the Einstein angle, $\theta_E = \sqrt{2m/L}$ (in our example the ring appears only for an instant, but this is unimportant). We want to see how this result emerges from the PGE. We will not actually perform any calculations in this section. Rather we will utilize the description of the fixed endpoint solution in the above section in conjunction with well known results and certain ideas of Pyne and Birkinshaw (1993) to arrive at a plausible understanding.

Briefly, we set up the mathematics of our lens system. For a Minkowski space background with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, equation (53) takes the form

$$x^{(1)\mu}(\lambda_2) = x^{(1)\mu}(\lambda_1) + (\lambda_2 - \lambda_1)\, k^{(1)\mu}(\lambda_1) + \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda)\, f^{(1)\mu}(\lambda)\, d\lambda \qquad (56)$$

An appropriate perturbation for our lens can be determined in the well known weak field approximation (see, e.g. Weinberg 1972, Chapter 10) with the result

$h_{\mu\nu} = \mathrm{diag}(-2\phi, -2\phi, -2\phi, -2\phi)$ with ϕ the Newtonian potential of the lens on the spatial hypersurfaces. It is easiest to suppose that both the emitter and the observer are sufficiently far from the lens that its gravitational field upon them may be neglected. For simplicity we could also assume that both the emitter and the observer have four-velocities given by $u^\mu = (1, 0, 0, 0)$.

Consider the one parameter family of null geodesics of our Minkowski space background given by

$$x^{(0)\mu}(\lambda) = \begin{pmatrix} \lambda \\ 2L - 2L\cos\theta - \lambda\cos\theta \\ 2L\sin\theta + \lambda\sin\theta \\ 0 \end{pmatrix} \tag{57}$$

At this point, there is nothing mathematically to stop us from constructing one null geodesic in the perturbed spacetime from each of these background geodesics, subject to the boundary conditions $x^{(1)\mu}(-2L) = 0$ and $x^{(1)i}(0) = x^{(0)i}(0)$. In addition, we could find $x^{(1)0}(0)$ by forcing our constructed geodesics to be null in $\eta_{\mu\nu} + h_{\mu\nu}$. These boundary conditions would ensure that each of the constructed null geodesics intercepts both q and the worldline of the observer. Finally, for each perturbed geodesic we could determine the angle it defined at the observer with the image of the lens.

Of course, our ability to construct so many such geodesics is an enormous warning sign: we know very well that at most two can be good approximations to the exact solution, one passing to each side of the lens. How then can we single out the two good approximate geodesics from the many bad ones? In Figure 2, we show three of the background paths we are considering, projected into the xy-plane. One, $x_C^{(0)\mu}$, travels diametrically away from the lens, one, $x_A^{(0)\mu}$, travels extremely close to the Schwarschild radius of the lens, R_s, and one is that background path which generates the best approximate perturbed path. The best approximation background path, $x_B^{(0)\mu}$, defines an angle with the x-axis at the emitter equal to θ_E. If we take seriously the cinematic description of our method offered in section 5, we should demand that any background path be suitable for an initial value calculation with initial value specified at the observer. This would immediately rule out the use of the path which passes very close to the lens (Pyne and Birkinshaw 1993). The other extreme path, however, is perfectly suitable for such a calculation so that this condition is not sufficiently stringent.

We could proceed by constructing the approximate paths associated to each of the unperturbed paths of Figure 2. Instead, we will use the analysis of section 5, above, to obtain important qualitative information about the solutions. For a given background geodesic, that analysis instructs us, first, to consider an initial value type solution, corresponding to a photon emitted backwards in time from the observer in the direction of the background geodesic. We know from Pyne and Birkinshaw (1993) that the initial value calculation returns a deflection at the lens equal to $L\alpha$, where α is the usual lens deflection angle appropriate to the background path used, $\alpha = 4m/L\sin\theta$. The method then would correct for the distance between $x^{(0)i}(0)$ and the origin, essentially by (vectorially) subtracting this distance from the deflection computed at the lens plane. The result, considered as the net linear deflection undergone by the photon, would then determine the position of the emitter on the observer's sky by applying to it the inverse Jacobi operator, basically $1/2L$. The paths constructed in this manner are shown in Figure 3.

Comparison of Figures 2 and 3, however, reveals an important distinction between the "correct" solution, x^μ_B, and the two extreme solutions, x^μ_A and x^μ_C. The two extreme solutions sample a lens potential totally unlike that felt by the background paths they were constructed from. In contrast, the solution marked x^μ_B feels essentially the same lens potential as that felt by the background geodesic it was constructed from, $x^{(0)}_B$. We could, for instance, conjecture that the two paths we seek are those generated by the two background paths which minimize

$$\left| \int_{x^\mu} |\phi|\, d\lambda - \int_{x^{(0)\mu}} |\phi|\, d\lambda \right| \tag{58}$$

where x^μ is the perturbed path associated to $x^{(0)\mu}$. Of course, (58) is mostly heuristic. We are not proposing this expression as a general error functional, assigning some significant real number measure of the error involved in using any particular background geodesic. * It does happen to suffice here, however.

* Nevertheless, such a functional should not be too difficult to find. The error in the perturbative geodesic expansion is a result of error in the approximate solution to Einstein's equation itself, which we do not consider, error from linearization of the Christoffel sysmbols and error from the truncation of the Taylor expansions used to express these quantities along the background geodesic (Pyne and Birkinshaw 1993). Simple matrix methods and the well known remainder term for Taylor's theorem can furnish crude bounds on these last two sources of error.

It is important to recognize that the above is exactly the sort of reasoning which must be applied in making rough consistency judgements in an initial value calculation. In fact, there is an analogous background path freedom in those calculations as well: by changing the initial data, an infinite number of background paths can be made to generate an infinite number of distinct perturbed geodesics all of which pass though a given point with the same tangent vector. Of course, there is a unique actual geodesic which passes through this point with the given tangent. It is nearly always assumed in perturbative calculations that the specific perturbed geodesic under investigation is, if not the best, at least an adequate approximation to the actual geodesic. In fact, we often have no more reason to expect this in the common initial value cases as we do for the fixed endpoint scenarios. It is a common hope, however, that forewarned is largely forearmed.

Suppose, now, that we have applied some error minimization and gained the usual lens solution. Would we have gained the usual lens time-delay formula? The answer is no, because the usual time-delay formula (see, e.g. Schneider, Ehlers, and Falco 1993, Chapter 4) contains a second order term, the geometric delay term, which is proportional to the square of the lens angle, itself a first order quantity. In fact, as we have already noted, the lens perturbation we are considering, when used in (10), immediately produces the Shapiro delay, but contributes nothing else. From a perturbation-theoretic point of view, the inclusion of the geometric delay term in the standard treatments is quite *ad hoc*. In Seljak (1994), for instance, the spatial components of the lensed photon path are solved for to linear order in the perturbing potential. This projected path is then "lifted" into the time domain by imposing $ds^2 = 0$ at second order. We do not intend this remark as a critique on the usual treatments: it is possible to perform the entire calculation at second order and show that only the usual terms are important. We bring this issue up in order to emphasize that there are astrophysical instances when *a priori* second order terms contribute numerically as importantly as first order terms. A nice example of this is the recent work of Frieman, Harari, and Surpi (1995).

7. Summary

We have presented an equation (29) for the time delay measured by two antennae observing a distant source through an adiabatic background of gravitational waves on Einstein-de Sitter spacetime. We have used the equation to determine the pattern of source motions on the sky induced by a background of gravitational radiation. We have also shown how these results may be extended to the curved FRW spacetimes. Our results are immediately applicable to situations, such as that considered here, for which the standard formulae of Kristian and Sachs (1965) do not apply, and so represent an important contribution to the theory of observations in perturbed spacetimes.

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References

Abbott, L., & Wise, M. 1984, Nucl. Phys., B244, 541

Carr, B. J. 1980, A&A, 89, 6

Einstein, A. 1916, *Preuss. Akad. Wiss. Berlin Sitzber.*, 688

Eubanks, T. M., & Matsakis, D. N. 1994, unpublished

Fabbri, R., & Pollock, M. 1983, Phys. Lett., 125B, 445

Frieman, J. A., Harari, D. D., & Surpi, G. C. 1995, preprint/a-ph9405015

Gradshteyn, I. S., & Ryzhik, I. M. 1994, Table of Integrals, Series, and Products (San Diego: Academic Press)

Gwinn, C. G., Pyne, T., Birkinshaw, M., Eubanks, T. M., & Matsakis, D. N. 1995, in preparation

Hawking, S. W. 1966, ApJ, 145, 544

Kristian, J., & Sachs, R. K. 1965, ApJ, 143, 379

Linder, E. V. 1988a, ApJ, 326, 517

Linder, E. V. 1988b, ApJ, 328, 77

Mukhanov, V. F., Feldman, H. A., & Brandenberger, R. H. 1992, Phys. Rep., 215 Nos. 5 & 6, 203

Pyne, T. & Birkinshaw, M. 1993, ApJ, 415, 459

Pyne, T. & Birkinshaw, M. 1995, ApJ, submitted

Romani, R. W., & Taylor, J. H. 1983, ApJ, 265, L35

Rubakov, V. A., Sazhin, M., & Veryaskin, A. 1982, Phys. Lett., 115B, 189

Sachs, R. K., & Wolfe, A. M. 1967, ApJ, 147, 73

Schneider, P., Ehlers, J., & Falco, E. E. 1993, Gravitational Lensing (Berlin: Springer-Verlag)

Seljak, U. 1994, ApJ, 436, 509

Taylor, J. H. 1992, in Proceedings of the 13th International Conference on General Relativity and Gravitation, eds. R. J. Gleiser, C. N. Kozameh, & O. M. Moreschi (Philadelphia: Institute of Physics Publishing)

Taylor, J. H. 1987, in Proceedings of the 13th Texas Symposium on Relativistic Astrophysics, ed. M. P. Ulmer (Singapore: World Scientific)

Taylor, J. H., & Weisberg, J. M. 1989, ApJ, 345, 434

Thompson, A. R., Moran, J. M., & Swenson, G. W. Jr. 1986, Interferometry and Synthesis in Radio Astronomy (New York: John Wiley and Sons)

Thorne, K. S. 1987, in Three Hundred Years of Gravitation, eds. S. W. Hawking & W. Israel (Cambridge: Cambridge University Press)

Weinberg, S. 1972, Gravitation and Cosmology (New York: John Wiley and Sons)

Figure 1. The geometry of Section 3. In the perturbed spacetime, light from a source at q travels along the null geodesic x^μ_A to an antenna at w'_A. The antenna is separated from the origin of spatial co-ordinates by the spatial vector $\vec{r}_A$. τ_A is the line of time translates of w'_A. The null geodesic of the background $x^{(0)\mu}_A$ joins the source to a point $w_A \in \tau_A$.

Figure 2. The background paths of Section 6, projected into the xy-plane. R_s is the Schwarzschild radius of the lensing mass. The arrows point in the direction of photon travel. θ_E is the Einstein angle of the lens.

Figure 3. The solution paths for the fixed-endpoint problem along the background paths of Figure 2. R_s is the Schwarzschild radius of the lensing mass. The arrows point in the direction of photon travel. θ_E is the Einstein angle of the lens.