Visualizing Electromagnetic Knots

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Maxwell’s equations admit strange solutions wherein any two field lines are linked (topologically). These are known as electromagnetic knots.

Rañada et al. recently introduced a topological theory of electromagnetism based upon these knots.
Faraday understood EM fields in terms of “lines of force”, now known as **field lines**.

Faraday’s intuitive understanding was later encoded mathematically in Maxwell’s equations.

Intuition and visualization breed scientific progress!

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]
Geometry

- Field line structure based on the **Hopf map**
  - Complex scalar field in space
  - Level curves are distinct circles
  - Surfaces of constant modulus are nested tori
Electromagnetic Knots

- Original construction of the fields cast in terms of differential geometry
- Initial solutions based on the Hopf map. Time-evolved solutions found by Fourier analysis
- The expressions are given by

\[ B = \frac{1}{2\pi i} \frac{\nabla \phi \times \nabla \bar{\phi}}{(1 + \bar{\phi}\phi)^2}, \]

\[ E = \frac{1}{2\pi i} \frac{\nabla \theta \times \nabla \bar{\theta}}{(1 + \bar{\theta}\theta)^2}, \]

\[ \phi = \frac{(ax - tz) + i(ay + t(a - 1))}{(az + tx) + i(a(a - 1) - ty)}, \]

\[ \theta = \frac{(ay + t(a - 1)) + i(ax + tz)}{(az + tx) + i(a(a - 1) - ty)}, \]

where \( a = \frac{1}{2}(r^2 - t^2 + 1). \)
Time Evolution of Field Lines

- We would like to visualize the time-evolved field lines while maintaining the topological structure induced by the Hopf map.
  - This is not always possible and requires the fields to behave in a certain manner.

- The electric and magnetic fields satisfy the "frozen field" condition
  - There exists a velocity field along which the field lines deform, maintaining their identity as such
  - Only possible because $E \cdot B = 0$ at all times.
The corresponding velocity field is given by

\[ v = \frac{E \times B}{E \cdot E} = \frac{E \times B}{B \cdot B} \]

Some surprising facts

- The velocity field depends functionally on \( z + t \)
- Any single element of a field line travels along a straight line at the speed of light.
In order to parametrize the field lines, we solve for the inverse of the Hopf map parametrically.

Time-evolved field lines are parametrized by deformation via the velocity field described previously.

The result is a simple parametric description of the field lines.
These images are based on a different construction of the knots involving complex fields.
Will be used in a colleague’s paper.
Prospects

- Gain a greater understanding of the nature of electromagnetism and EM knots.
- Apply our understanding to related areas, such as complex fields, complexified spacetime, and twistor theory.