University of California
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Convection-Modified Dynamo Causes Enhanced Turbulence in Accretion Disks

A dissertation submitted in partial satisfaction of the requirements for the degree

Bachelors of Science
in
Physics

by

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June 2016
The Dissertation of Evan Yerger is approved.

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June 2016
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Evan Yerger
Acknowledgements

I would like to first and foremost thank my advisor, Dr. Omer Blaes. He has been an incredible mentor to me and without many of the freedoms he has afforded me, I would not have cultivated the love for plasma physics that I have today.

Matt Coleman has also been an incredible source for knowledge, advice, and support; without his consistent guidance I would not have been nearly as effective.
Abstract

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by

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Accretion disks are extremely ubiquitous throughout the Universe and characterize a large amount number of phenomena in the night sky; yet, they are still not well understood. As steady-state disks provide offer little insight, we investigate a class of non steady-state accretion disks in dwarf novae that reveal to us quite a bit about their structure through their quasi-periodic outburst cycle.

Throughout the outburst phase of these disks, large-scale magnetic fields are found to oscillate quasi-periodically. However, towards the end of the outburst phase, the disk becomes unstable to convection and the parameterized disk stresses increase. This increase is accompanied by a change in the large-scale fields: they no longer oscillate.

Large-scale magnetic fields - especially in accretion disks - arise from correlated turbulence within the disk. Thus a modification to the large-scale structure indicates a change in small-scale turbulence. By investigating how the change in large-scale field couples to the small scale, we have a unique window into the turbulence in accretion disks, and this technique has paid dividends. I argue a few things can be gathered. There exist two separate dynamos within the accretion disk: one coupled to hydrodynamics and the other magnetically-dominated. The former dynamo exhibits an instability which results in field reversals that propagate outward from the midplane region, causing the latter dynamo to switch signs of field as well. During convection, the nature of the causality switches in that the magnetically-dominated dynamo feeds into the hydrodynamic dynamo, suppressing field reversals.
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Chapter 1

Introduction

Accretion disks are perhaps one of the most fundamental phenomena in the Universe, playing a quintessential role in everything from planet formation to the brightest objects we observe in the night sky. The essential idea of an accretion disk is that, much like the Earth orbiting the Sun, any object orbiting around a central mass conserves its angular momentum as it does so. Therefore, matter cannot simply ‘fall into’ an object it is orbiting around on its own - it must interact with other matter to exchange angular momentum. Accretion disks are astrophysical objects in which this momentum exchange can happen at very high rates due to enormous amounts of friction within the disk. The higher the levels of friction within the disk, the more matter it accretes and the hotter (brighter) it gets.

Molecular viscosity was first suggested as a method of angular momentum exchange within accretion disks; however, it is much too small to explain the observed rate of energy release from these disks, which we interpret through their brightness, or luminosity. Clearly, there are other processes within these disks that facilitate the accretion of matter much more substantially than viscosity alone. However, it is quite difficult to image accretion disks at the level necessary to resolve these processes, so we end up parameter-
izing the stresses in the disk with a dimensionless parameter called alpha. Unfortunately, for steady-state disks (which characterize the vast majority of accretion disks), alpha is incredibly difficult to measure accurately.

Serendipitously, astronomers have inadvertently been observing non-steady-state accretion disks for over one hundred years. These objects were originally called Variable Stars due to their strange behavior, characterized by quasi-periodic luminous outbursts. It was later determined that these Variable Stars were actually a binary star system in which two stars - one a White Dwarf (very small and dense), the other a late-term star (cool and bloated) - orbit each other in close enough proximity that the dense star begins to pull matter off of the surface of the other star. Since the stars are orbiting about their shared center of mass, the matter falling toward the dense star has angular momentum about the star, and forms an accretion disk. These star systems have since been named more appropriately as Dwarf Novae.

The outburst-quiescence cycle of Dwarf Novae, which correspond to high and low alpha respectively, are a result of an instability caused by the principle matter being accreted - hydrogen. The instability, which will be explained in more detail in section 2.1, is essentially explained by the fact that hydrogen ionizes in the middle of the temperature-density spectrum defined by the quiescent and outburst phases of the accretion disk. Starting in the quiescent state, the disk is relatively cool, sparse, and the hydrogen is unionized. As matter is continuously added to the disk, it eventually becomes dense enough that hydrogen rapidly ionizes, directly resulting in a sharp increase in opacity.

Hydrogen ionization also results in the accretion disk becoming highly conductive, such that magnetic fields are able to thread through the fluid like rubber bands. This state of matter is referred to as a plasma. Magnetic fields in the plasma are responsible for turbulence that facilitates angular momentum redistribution within the disk and explains the high alpha values associated with the outburst phase. These fields essentially act like
a spring between adjacent particles in the disk, increasing the angular momentum of one particle at the expense of the other. In order to maintain a stable orbit, the lower-momentum particle falls inward toward the White Dwarf while its counterpart is ejected outward.

The dynamics of plasmas are highly nonlinear, so outside of some uninteresting and reductive situations, there exist no analytical solution to the equations of motion. This means that plasmas must be simulated by computer and makes explaining exactly how magnetic fields work to redistribute angular momentum an extremely difficult task. The qualitative method described in the previous paragraph actually explains a method discovered in 1991\[1\] by Steven Balbus and John Hawley called the Magnetorotational Instability. While this method explains quite a bit about the small scale turbulence that arises in simulations of accretion disks, it does not explain the emergence of large-scale magnetic field structures which also play a role in enhancing the stresses (and thus momentum redistribution) within the disk.

Large-scale magnetic field structures are generally the result of a process in which the field folds back on itself in a self-reinforcing process called a dynamo. During most of the Dwarf Novae outburst phase, the disk is still transparent enough such that radiation is able to effectively transport thermal energy from the center of the disk (the result of turbulent and magnetic dissipation driven by turbulence) outside the photosphere\[1\]. When transport is dominated by radiation in this way, magnetic fields exhibit an oscillatory pattern, resulting in a ‘butterfly diagram’. Towards the end of the outburst phase, radiation cannot dissipate energy quickly enough and the accretion disk becomes unstable to convection, where thermal energy is transported by bulk motions of energy-dense fluid instead of photons. During these convective epochs, the large-scale magnetic fields no

\[1\]The photosphere is the optical edge of the disk, where one looking at the disk would see a solid object start. It is characterized by the point where the mean free path of a photon from that point is infinity, i.e. it is no longer further impeded by matter.
longer show an oscillatory pattern and instead 'lock' in place.

## 1.1 Accretion Disks

Accretion disks are a ubiquitous phenomena throughout the Universe, characterizing protoplanetary disks, many binary systems, and active galactic nuclei (AGN). Protoplanetary disks are the lowest-energy manifestation of an accretion disk and are usually found around newly-formed stars. By contrast, those around AGN (supermassive black holes) are the most energetic and can form quasars - the most luminous objects in the Universe.

### 1.1.1 Basic Physics and Properties

Here I will include a synopsis of the basic physics of an accretion disk and will elucidate aspects of its structure, starting with the first developments in accretion disk physics and detailing issues along the way. I will essentially follow descriptions in [2], [3], and [4]. Consider a cloud of gas in arbitrary bound orbit around a central object. If it is assumed that the gas can radiate energy easily (so that it not retained within the gas and escapes), then the effect of molecular collisions and other processes within the gas will be to reduce the energy of each molecule to its lowest possible state, while still retaining its angular momentum. Collisions will also have the effect of reducing any complex three-dimensional gas orbit into a two dimensional circular, planar orbit with both the plane and direction of rotation defined by the net angular momentum of the initial orbit.

In this thesis I will work with axisymmetric, geometrically thin accretion disks such that $H/R \ll 1$ where $H$ is the scale height of the disk and $R$ is the radius. This is useful because many approximations for accretion disks are dependent on expansion in $H/R$ and we can reliably work in a cylindrical coordinate system $(R, \phi, z)$ where $z = 0$
is defined by the axis of symmetry in the plane of the disk. The surface where \( z = 0 \) is commonly referred to as the midplane. In a thin disk, the gravitational force of the central object on the fluid will be balanced by the centrifugal force of its acceleration:

\[
v^2_\phi/R = -\partial \Phi / \partial R \quad \text{where} \quad v_\phi \text{ is the tangential orbital velocity and} \quad \Phi = -GM/R \text{ is the gravitational potential of an object of mass } M \text{ at radius } R.
\]

If the self-gravity of the disk can be ignored, the orbital angular velocity profile \( \Omega(R) \) will therefore be Keplerian with \( \Omega = \sqrt{GM/R^3} \). Assuming the disk is in hydrostatic equilibrium with the force of gravity from the central object, i.e. that the vertical pressure gradient \( \partial_z P \) is balanced by the linearized gravitational force about \( z = 0 \), we have \( \partial_z P = -GMz/R^3 \). Upon integrating this relationship across the vertical extent of the disk, we find that the density \( \rho \) as a function of \( z \) and \( R \) is

\[
\rho(z, R) = \rho_c(R) \exp\left(-\frac{GMz^2}{2c_s^2R^3}\right),
\]

where \( c_s \) is the isothermal sound speed defined by \( c_s^2 = \rho/P \). Given the local density follows a Gaussian distribution, we now have a natural definition for the scale height \( H \) as \( H^2 = c_s^2R^3/GM \), which is analogous to the definition of the standard deviation of a Gaussian probability distribution.

Although the gas in accretion disks has a way to rid itself of excess energy through radiation, there is no analogous mechanism for angular momentum, so the total angular momentum of an accretion disk is always conserved (in the absence of external torques). So, for a disk to accrete matter, it must exchange angular momentum between neighboring elements in the disk. Assuming that the gas in accretion disks is dynamically similar to gas here on earth, any non-uniformity in the gas’s velocity \( v \) such that \( \partial_i v_i \neq 0 \), will

\[\text{2}^{\text{The vertical extent of the disk is defined to be from } z = -\infty \text{ to } z = +\infty.}\]

\[\text{3}^{\text{Here } i \text{ refers to any Cartesian coordinate direction and the Einstein summation convention is not assumed. In all other expressions, however, the convention will be used.}}\]
produce a shear stress \( S_i = \mu \partial_i v_i \) where \( \mu \) is the dynamical viscosity of the gas. The effect of this stress is to exchange momentum between two differentially moving fluid elements, since the faster element will be ‘pulled back’ by the slower and visa-versa. Simplifying this picture for an accretion disk, we define an annulus of the disk to be an infinitesimally small ring of thickness \( dR \) at constant radius. Taking the annulus to be infinitely tall, its mass will be \( 2\pi \Sigma R dR \), where \( \Sigma \) is the surface density of the disk defined as \( \Sigma(R, \phi) = \Sigma(R) = \int_{-\infty}^{+\infty} \rho(z, R) dz \), where \( \rho(z, R) \) is from equation 1.1 and the dependence on \( \phi \) is dropped due to axisymmetry.

Kinematic viscosity dissipates the kinetic energy in shearing flows as heat. Assuming that the accretion disk in question is in a steady state (or at least evolving on a timescale much longer than the human observational timescale), the inward mass flux at any annulus is constant: \( \dot{M} = -2\pi \Sigma R v_R = \text{constant} \). The viscous dissipation rate \( D(R) \) per unit time per unit area for an annulus at radius \( R \) is

\[
D(R) = \frac{1}{2} \nu \Sigma(R \partial_R \Omega)^2 = \frac{3GM\dot{M}}{4\pi R^3} \left[ 1 - \left( \frac{R}{R_*} \right)^{1/2} \right], \tag{1.2}
\]

where the second equality holds for a Keplerian disk around a central object with radius \( R_* \) (see Pringle 1981[2] for a detailed derivation). The dependence of dissipation on viscosity has vanished because we have implicitly assumed - as a result of our assumption of constant mass flux - that viscosity scales with surface density in a way that holds this mass flux constant. The total disk luminosity is therefore

\[
L_{\text{disk}} = 2\pi \int_{R_*}^{\infty} D(R) R dR = \frac{1}{2} \frac{GM\dot{M}}{R_*}. \tag{1.3}
\]
1.1.2 Anomalous Viscosity and the Alpha Prescription

The model of accretion disks presented in the previous section proved to be incredibly problematic. For one, inserting a realistic value for the kinematic viscosity into the theory predicted an incredibly low accretion rate and a luminosity orders of magnitude lower than what had been observed. Somehow, the viscosity of (especially luminous) accretion disks was ludicrously higher than was predicted - leading researchers to coin the term ‘anomalous viscosity’. To make matters even worse, direct measurement of the viscosity parameter proved to be incredibly difficult since it falls out of the equation for luminosity. The best that could be done was to measure the luminosity and try to solve for $\nu$ given estimates for the other parameters in 1.2; however, the uncertainties in these parameters compounded such that the uncertainty in $\nu$ itself was so large that it gave very little insight into its value. Given the large number of unknowns and the wide range of experimental predictions for $\nu$, researchers were only left to speculation. In 1973, Shakura and Sunyaev introduced the alpha prescription in which all of the uncertainties in the anomalous viscosity were compounded into a single dimensionless parameter $\alpha$. Some authors proceeded by modeling the viscosity as an effective turbulent viscosity of the form $\nu_{\text{eff}} \sim l \cdot v$ where $l$ and $v$ represent the size and turnover velocities of the largest turbulent eddies. Further making the (quite justifiable) assumptions that the eddies are smaller than the scale height $H$ and that the turbulence is subsonic, we arrive at the popular expression for the effective kinematic viscosity:

$$\nu_{\text{eff}} = \alpha c_s H,$$

where $\alpha \leq 1$ to encapsulate the assumptions.

It has been pointed out, however, that this particular manifestation of the alpha prescription is potentially inaccurate and misleading because it assumes that the tur-
bulence in accretion disks mathematically behaves like a viscosity by coupling it to the shear rate. At the least this prescription leads to wrongful intuition about the turbulent stresses within the disk and at worst isn’t actually physical. A much more realistic and general manifestation of the alpha prescription\textsuperscript{4} - and the one that will be used henceforth in this thesis - is of the original form given by Shakura and Sunyaev where \( \alpha \) parameterizes vertically-averaged internal stress\textsuperscript{5} \( \omega_{r\phi} \):

\[
\omega_{r\phi} = \alpha P.
\] (1.4)

Here \( P \) is the vertically-averaged pressure. Incorporating a viscosity and stress term into the momentum balance equations, we see that phenomenologically the dissipation due to viscosity is nearly negligible when compared to that of stresses in the disk. So, as a quite accurate approximation, the viscous term can be left out of the equations, leaving only the stress terms to contribute.

### 1.2 Dwarf Novae to the Rescue

Although the altered \( \alpha \) prescription presented in \textsuperscript{1.4} is of a different form than the anomalous viscosity prescription, it still suffers from the same issues in that it is incredibly hard to estimate \( \alpha \) from steady state disks. Thankfully, there exist a class of dynamical accretion disks that are fairly large in number and quite easy to observe which offer a much more reliable experimental determination of \( \alpha \) than their more invariable cousins. These dynamical disks play a behavior-characterizing role in the larger astrophysical objects in which they are a constituent, named dwarf novae. Dwarf novae are one of the

\textsuperscript{4}I include the previous, incorrect description of the alpha prescription not to confuse the reader, but partly for pedagogical continuity and partly because this formalism is still heavily used within astrophysics and I wish to be explicit in the formalism I am using.
most-observed objects in the night sky by amateur and professional astronomers alike for their intense, episodic luminous outbursts. Perhaps the most famous example of a dwarf novae is a variable star named SS Cygni.

![SS Cygni Light Curves](image)

Figure 1.1: Light curves for SS Cygni from September 27, 1896 - April 7, 1992. The x-axis is time (years are labeled) and the y-axis is the apparent magnitude, which scales as the log of the luminosity. Consequently, SS Cygni in outburst is roughly a factor of 40 times more luminous than in the quiescent state. The length of outbursts is roughly a week or two, to give a temporal scale.

SS Cygni was discovered in 1896 by E.G. Pickering and was noted for its incredibly luminous, quasi-periodic outbursts. The star so enchanted astronomers that its light curve - its luminosity plotted over time - has been continuously recorded since its discovery.

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5The term variable star simply means that the luminosity of the star fluctuates over time. In this case we will see that it’s a bit of a misnomer since dwarf novae are binary star systems, but astronomers have never really had a good track record naming things, so we’re stuck with it.

6For more information visit [https://www.aavso.org/vsots_sscyg](https://www.aavso.org/vsots_sscyg)
covery. From a sample of the novae’s light curves shown in figure 1.1, it is easy to see what we mean by quasi-periodic outbursts. Most of the time the star rests at relatively low luminosity in what’s called its quiescent state, the luminosity will sometimes quickly grow by a factor of four and remain radiating at that intensity for some while in its outburst state. These outbursts will only last for a week or two, at the end of which the star will retire into its quiescent state for some four to ten weeks. All in all, the star is in outburst for about a fourth of its life.

SS Cygni, like all other dwarf novae, is not a single star, but in fact is binary star system in which a red dwarf-type star and a white dwarf star tightly orbit their common center of mass about 100,000 miles apart\(^7\). Their orbit is so close that the sphere of gravitational influence, or Roche lobe, of the white dwarf permeates the surface of its companion and gravitationally strips off matter from the star. Because the stars are rotating (orbiting), the matter falling into the white dwarf has considerable angular momentum about the star and forms an accretion disk. It is understood that the quasi-periodic outbursts of SS Cygni are due to an instability in the accretion disk itself. This important fact means that white dwarf binaries offer the most accurate estimates for \(\alpha\) that exist for accretion disks\(^6\). Measurement of the relative luminosity and duration of each phase along with the timescale of the transition between phases gives an empirical value\(^7\) for \(\alpha \sim 0.1\) during outburst. Other calculations\(^8\) reveal that \(\alpha\) in the quiescent state is about an order of magnitude smaller.

To see how this all fits together, it is important to understand that the outburst and quiescent states are stable thermodynamic equilibria and the transition between the two is mediated by an instability (see section 2.1 for a more detailed explanation). Equilibria are by definition a balance in which the rate of heating within the disk is equal to the rate of thermal transport out of the disk. This means that the net energy inside the

\(^7\)This corresponds to one one-thousandths the distance at which Earth orbits the sun.
disk is constant and correspondingly so too is the temperature. High relative values of $\alpha$ in the outburst state indicate that the internal stresses are also much higher. Higher stresses dissipate more heat in the disk which, in order to maintain equilibrium, leads to the disk radiating much more energy resulting in high luminosity we observe. But what, then, is responsible for the high stresses that allow for a stable (at least for a little while) outburst state in the first place?

Higher stresses during outburst are the result of the fact that hydrogen, at the temperatures and densities found in the outburst phase, is ionized and forms a plasma. Plasmas conduct electricity with little resistance and as a result can be threaded with magnetic fields. Since moving magnetic fields induce currents and currents again induce magnetic fields, any movement of the plasma will result in the magnetic field evolving in a non-trivial way. The principle equation that governs the dynamics of plasmas is the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),$$

which will be used extensively in many different explicit forms throughout this work.

1.3 Overview

In the chapters that follow, I will cover in more detail the relevant features and concepts necessary to understand the conclusions of this thesis. Starting in chapter 2 I formally introduce dwarf novae, their outbursts, and the related phenomenon of convection. I then continue with plasma physics basics in chapter 3 and move into the simulations used to understand accretion disks in outburst and formally introduce the behavior that prompted the main work of this thesis, which is found in chapters 5 and 6.
Chapter 2

Dwarf Novae

In this chapter, I formally introduce the disk instability model (DIM) and ....

2.1 The DIM and Luminous Outbursts

The outburst-quiescence cycle of dwarf novae are well described by the disk instability model\textsuperscript{[9]} (DIM) - an outline of which is presented here. As explained previously in section 1.2, this cycle is a result of an instability within the accretion disk itself. Outbursts are a result of large stresses generated within the disk that create high amounts of heating and thus relatively high luminosities. The quiescent state is largely the opposite: disk stresses are relatively low and so heating and luminosity are also relatively low. Although the states are observationally quite different, they share the same characteristic: both are thermodynamic equilibria in which the rate of stress heating is equal to the rate of cooling, or the flux of energy escaping from the disk $J$. This equality does not hold in the transition between states and results in a rapid change in temperature. Transition between the quiescent state and outburst state is accomplished via a thermodynamic instability in which heating exceeds cooling, causing the temperature to rise very quickly.
The transition back to quiescence is unsurprisingly mediated by an instability where cooling exceeds heating. The source of this instability is the dependence of disk’s opacity on temperature and density.

2.1.1 A Brief Aside on Opacity

The opacity of a gas $\kappa$ is the degree to which electromagnetic radiation is absorbed by a medium. However, this is not a simple problem. A photon interacts with matter via the mechanisms of electron scattering, free-free absorption, bound-free absorption, and bound-bound absorption\[1\]. Each of these mechanisms, moreover, will be dependent on the atomic makeup and density of the gas in question as well as the frequency of the photon, making opacity a very complicated process to model. For this reason, there are two main averaging procedures that define mean opacities, but each utilize a different weighting function. The Planck mean opacity $\kappa_P$ uses the normalized Planck black body radiation energy density spectrum while the Rosseland mean opacity $\kappa_R$ uses the temperature derivative of the Planck distribution as the weighting function. The Planck mean opacity is primarily used to describe the interchange between radiation and internal energy density within a gas while the Rosseland mean is used in the calculation of the force due to radiation pressure. Plots of the Rosseland and Planck mean opacity as a function of density and temperature for a gas with a solar composition of elements\[2\] from Hirose et. al 2014[6] are presented in figure (*****).

Planck and Rosseland mean opacities therefore often characterize the thermal and pressure equilibrium of astrophysical gases, respectively, and play a large role in accretion disk physics. Stress dissipation converts mechanical energy into heat $J$, which defines

\[^1\]I will not go through the details of each here. For a more thorough explanation, see ********source********.

\[^2\]Solar composition simply means that the relative densities of different elements matches that found in the sun. This distribution of elements is likely to be found in dwarf novae accretion disks.
a temperature. Assuming the gas radiates like a black body, this temperature will be defined by the Stefan-Boltzmann law:

\[ J = \sigma T^4, \]  

(2.1)

where \( \sigma \) is the Stefan-Boltzmann constant. This temperature will define a thermal pressure \( p \) through a thermodynamic equation of state (EOS) as well as a radiation pressure \( P \) through the Rosseland mean opacity. The efficiency by which thermal radiation can escape the disk is governed by the Planck mean opacity and is largely responsible for characterizing the thermodynamic equilibria of accretion disks. We will see how this plays out in the next section.

### 2.1.2 Back to Luminous Outbursts

Accretion disks generally have large density and temperature variations across their vertical extent. The outburst-quiescence cycle is a global phenomenon, so we need a way to characterize the disk on large (i.e. nonlocal) scale, which we will accomplish by considering the behavior of annuli. The annulus density will be given by the surface density \( \Sigma \) (defined in section 1.1.1) and an effective temperature \( T_{\text{eff}} \), which is again defined by the Stefan-Boltzmann law:

\[ J_{\text{surf}} = \sigma T_{\text{eff}}^4, \]  

(2.2)

where \( J_{\text{surf}} \) denotes the energy flux measured at a surface parallel to the disk midplane. The flux through this surface captures the total power radiated away from the disk and,

---

3 Note the calculation of both pressures depend explicitly on the gas density \( \rho \).

4 These variations are also present across the radial extent of the disk; here we are just considering the disk at a specific fiducial radius \( R \).
assuming thermodynamic equilibrium, is equal to the rate of energy injection by stress heating.

![Figure 2.1](image.jpg)

Figure 2.1: A plot of effective temperature vs. surface density, detailing the outburst-quiescence cycle of dwarf novae. Points a-c define the quiescent state, where the disk is gains net mass until an ionization instability is hit at c in which the gas becomes more opaque as temperature increases. Points d-f correspond to the outburst phase, which ends in the opposite instability defined between c and d, where cooling > heating. Figure from Lasota 2001.

The outburst-quiescence cycle can now be summarized by a plot of effective temperature vs. surface density, shown in figure 2.1. For specificity, I have labeled points and curves of interest by the letters a through f. We begin at the point labeled a, where we are in the quiescent state just after an outburst. Recall that $\alpha_q$ is an order of magnitude lower than $\alpha_o$ (the subscripts q and o denote quiescent and outburst, respectively). $\alpha_q$
is in fact so low that the rate of mass inflow from the red dwarf-type companion star exceeds the accretion rate, causing the surface density of the disk to increase along the curve labeled b. A balance between heating and radiative cooling is struck along the curve until point c, where the opacity begins to increase with an increase in temperature. This causes a runaway effect (instability) where the effective temperature of the disk increases rapidly until a new thermodynamic equilibrium is reached at point d - the entry point to the outburst phase. Massive stresses within the disk increase the accretion rate to an extent higher than the mass transfer rate from the companion star and the disk begins to shed its surface density. Along the outburst path e, a large temperature gradient arises between the interior of the disk, where most of the stress heating occurs, and the photosphere, where radiation can freely propagate out of disk.

As the disk draws nearer towards the end of the outburst state (f), large temperature gradients begin to form as a result of an increase in disk opacity. These gradients can become so large that the disk becomes unstable to convection \[6\] (this will be detailed in the next section 2.2). When this occurs, transport of energy from the center of the disk outward is dominated by advection of high-internal energy cells instead of radiation. The result is that the local temperature within the disk drops considerably (roughly in half) and the gas becomes incredibly opaque, creating an even steeper temperature gradient and locking in convection. The bulk motions created by convection further enhance the stresses and further rarefy the disk. Eventually the disk becomes so diffuse and cold (point f) that opacity begins to decrease as temperature is decreased and another thermodynamic instability is reached, causing the disk to radiate more energy than is injected by heating. As this heat is lost the disk’s effective temperature tumbles until the quiescent equilibrium is achieved again at point a.
2.2 Convection

In short, convection is a thermodynamic instability caused by a large temperature gradient\textsuperscript{5} characterized by Schwarzschild’s criterion for the convective instability in a homogeneous fluid. I will mainly be following the derivation in Shu, The Physics of Astrophysics (1991). Consider a vertically-stratified gas heated from below such that the temperature gradient $\partial T/\partial z < 0$. Select a blob of gas from the gas; we will assume that it is in thermal and mechanical equilibrium with its surroundings. The stability of the blob will be determined by giving it a small vertical displacement. If the blob experiences a restoring force to its initial position, it is stable; however, if the blob is accelerated away from the equilibrium position, it is unstable. During this perturbation, we assume that the blob will maintain pressure equilibrium by expanding and contracting adiabatically (constant entropy $s$), that is with no heat transfer between it and the surrounding fluid\textsuperscript{6}.

Displacing the blob downward, we find that its density will vary as

$$ (\partial \rho)_{\text{blob}} = \left( \frac{\partial \rho}{\partial P} \right)_s dP $$  \hspace{1cm} (2.3)

where the subscript on the derivative indicates constant entropy and $P$ is the pressure of the external medium. The density of the external medium will vary as

$$ (\partial \rho)_{\text{ext}} = \left( \frac{\partial \rho}{\partial P} \right)_s dP + \left( \frac{\partial \rho}{\partial s} \right)_P ds. $$  \hspace{1cm} (2.4)

The motion of the blob will be unstable if it is overdense relative to its surroundings: $(\partial \rho)_{\text{blob}} > (\partial \rho)_{\text{ext}}$. Our assumption that the blob is in pressure equilibrium allows us to set the two $dP$s in equations (2.3) and (2.4) equal. After imposing the density inequality

---

\textsuperscript{5}I will quantify exactly what I mean by 'large' momentarily.

\textsuperscript{6}If the blob is small, the time it takes to normalize the pressure will be much less than the heat diffusion timescale.
required for the instability, we find that the medium will be convectively unstable if

\[
\left( \frac{\partial \rho}{\partial s} \right)_p ds < 0
\]  

(2.5)

for a downwards displacement. This equation is not very intuitive, so using various thermodynamic identities, we find Schwarzschild’s criterion for a convective instability:

\[
ds > 0 \text{ in the direction of gravity.}
\]  

(2.6)

A gradient for which \( ds = 0 \) is called an adiabatic gradient, so any gradient for which \( ds > 0 \) (in the direction of gravity) is called superadiabatic and any gradient for which \( ds < 0 \) is called subadiabatic.
Chapter 3

The Physics of Plasmas

The thermodynamic equilibrium of the outburst state is highly dependent on the relatively large stresses that are associated with it. Although it’s well understood that turbulence must be responsible, it is still a bit of an open question as to how, exactly, the turbulent stresses arise. At first glance, one may be inclined to think that purely hydrodynamical turbulence may be the culprit. There are, however, some issues with this. Consider a uniformly orbiting fluid particle in an arbitrary rotation profile \( \Omega(R) \). A particle will experience the centrifugal force of gravity, centripetal force of acceleration, and Coriolis force in its co-moving frame of reference. Expanding the forces about a constant radius in \( x \) and \( y \) (corresponding to displacements in \( R \) and \( \phi \), respectively) we have a coupled set of second order partial differential equations. Assuming a solution of the form \( \exp(i\omega t) \) we find

\[
\omega^2 = R\partial_R \Omega^2 + 4\Omega^2 \equiv \kappa^2, \quad (3.1)
\]

where \( \kappa \) is the epicyclic frequency.

For Keplerian profiles, \( \kappa^2 = \Omega^2 > 0 \), so the frequency is real and the equation of
motion is oscillatory. Particles in Keplerian orbital profiles are always stable to linear perturbations and will eventually de damped out\textsuperscript{1} Clearly, any excitation of hydrodynamical turbulence within an accretion disk would not be self-sustaining and this is inconsistent with our observations. Note that stability to linear perturbations is very different than nonlinear stability, which still remains an open question at astrophysically-relevant Reynolds numbers
\[ Re = \frac{UL}{\nu}, \] (3.2)
where $U$ and $L$ are velocity and length scales, respectively\textsuperscript{2}. However, we will see that this issues is not much of a concern.

3.1 Basic Plasma Physics

The self-sustaining nature of accretion disk turbulence is explained by the fact that the elemental species in the disk are highly ionized at the high temperatures associated with the outburst state\textsuperscript{3} Ionization so changes the properties of a gas that it is considered to be a phase change into a new state of matter called a plasma. When ionization occurs, the electrons that previously resided within the electric potential well of their host nuclei become energetic enough to break free. Therefore, plasmas are really a collection of relatively light, negative electrons and heavy, positive ions that are free to move independently of one another. This characteristic allows plasmas to be highly electrically conductive. In the regimes under present consideration (at least in outburst) the electrical resistivity $\sigma$ of the plasma is so negligibly small, we will approximate $\sigma = 0$

\textsuperscript{1}Although damping terms were not explicitly included in the derivation of equation 3.1 the result would be an imaginary part to the frequency that would exponentially damp the oscillatory mode we assumed as a solution.

\textsuperscript{2}$U$ and $L$ are chosen to be representative of the dynamics at hand; $U$ can be taken to be the sound speed $c_s$ (or Alfvén speed in the case of 3.3) and $L$ the scale height of the disk.

\textsuperscript{3}There is still ionization during the quiescent state, but to a much lesser extent.
The Physics of Plasmas Chapter 3

for the rest of this thesis. Defining the magnetic diffusivity - an analog to kinematic viscosity- as \( \eta \equiv c^2/4\pi \sigma \), the magnetic Reynolds number

\[
R_m = \frac{UL}{\eta}, \quad (3.3)
\]

where \( c \) is the speed of light in a vacuum\(^4\) is formally equal to infinity. Those new to this area of research may be disconcerted by this seemingly unphysical approximation; however, this assumption is one of the fundamental assumptions of ideal magnetohydrodynamics (MHD), which is used extensively in astrophysics and throughout this thesis. For an excellent introduction to MHD, see Essential Magnetohydrodynamics for Astrophysicists by H.C. Spruit.

### 3.1.1 Equations of Motion

The assumption of infinite electrical conductivity in ideal MHD means that the electric field must be zero everywhere inside the plasma. Any electric field will produce an infinite current that will immediately short the source of that field\(^5\). This condition, however, is only valid in the rest frame of the plasma. For any other non co-moving observer, there will be an electric field to account for that will induce a magnetic field. The dynamics of this process are captured by the induction equation

\[
\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (3.4)
\]

where \( \mathbf{v} \) is the velocity of the fluid relative to the observer. In regular electrodynamics, the magnetic field is divergence-free: \( \nabla \cdot \mathbf{B} = 0 \). Taking the divergence of the induction

\(^4\)We assume that the speed of light is not significantly altered in plasmas, as it may be in other materials.
\(^5\)We assume the plasma is electrically neutral and remain so due to charge conservation.
equation 3.4, we find that $\partial_t (\nabla \cdot B) = 0$ and so the divergence-free condition does not need to be considered explicitly when solving the MHD equations so long as the initial conditions satisfy the condition. Assuming all fluid velocities are nonrelativistic, the current $j$ is given by

$$j = \frac{c}{4\pi} \nabla \times B. \quad (3.5)$$

If we take the divergence of the above relationship, we find that $\nabla \cdot j = 0$ so currents in MHD are also divergence-free; however the utility of currents in understanding the dynamics of MHD is not nearly as great as in regular electrodynamics. The Lorentz force $F_L$ is

$$F_L = \frac{1}{c} j \times B = \frac{1}{4\pi} (\nabla \times B) \times B. \quad (3.6)$$

Momentum balance is most generally given by

$$\rho \frac{d\mathbf{v}}{dt} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \frac{1}{4\pi} (\nabla \times B) \times B + \rho g. \quad (3.7)$$

Here $g$ is the acceleration due to gravity. Finally, the constraint of mass conservation is ensured by the continuity equation

$$\frac{d\rho}{dt} + \nabla (\rho \mathbf{v}) = 0. \quad (3.8)$$

### 3.1.2 Properties of Plasmas

In MHD it is extremely useful to conceptualize the magnetic field in terms of field lines. Given a magnetic field vector $\mathbf{B}(x, y, z, t)$ that is consistent with equations 3.4 and 3.8, we define field lines as paths that are everywhere tangent to $\mathbf{B}$ at some instant in time. The induction equation 3.4 can therefore be understood as the evolution equation for magnetic field lines under the influence of a fluid flow. Flows purely parallel to magnetic
field lines have no effect on them since the cross product of two parallel vectors is 0. However, flows with a perpendicular component to the magnetic field (with either having some spatial variation) do have an effect and can alter the magnitude and direction of $\mathbf{B}$ or, equivalently, the density and direction of field lines. This behavior is a consequence of ideal MHD known as the flux freezing condition in which field lines are considered to be perfectly advected with the flow without slipping.

The forces created by magnetic field lines are given by the Lorentz force and influence the flow through the momentum equation. The extent to which the evolution of the flow is governed by hydrodynamic or magnetic forces is encapsulated by a parameter called the plasma beta:

$$\beta = \frac{8\pi p}{B^2} \equiv \frac{p_{\text{gas}}}{p_{\text{mag}}},$$  \hspace{1cm} (3.9)

the ratio of gas to magnetic pressure. Consider a cylindrical volume enclosing a constant density of straight field lines such that there is zero magnetic flux through the sides of the cylinder, i.e. $\mathbf{B} \cdot \mathbf{n}_{\text{side}} = 0$, where $\mathbf{n}_{\text{side}}$ is the unit vector normal to the curved face of the cylinder with no field lines present outside the cylinder. The Lorentz force can be rewritten in the form

$$\mathbf{F}_L = -\frac{1}{8\pi} \nabla B^2 + \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B}$$  \hspace{1cm} (3.10)

and taking the regular cylindrical coordinates $(r, \phi, z)$ we find a force exerted on each cap of the cylinder $F_z = -B^2/8\pi$ and a force exerted on the curved face $F_r = B^2/8\pi$. Clearly, magnetic fields exert both a pressure ($F_r$) and a tension ($F_z$). For the cylinder

---

6 In non-ideal MHD where we consider a finite resistivity, field lines can 'slip' and diffuse through the plasma. The extent to which this slipping governs the dynamics of the plasma is captured in $Rm$ (equation 3.3) with inductive processes dominating for $Rm \gg 1$ and diffusive processes dominating for $Rm \ll 1$.

7 Assume the cylinder is infinitely long and straight and we are considering a finite length of that cylinder. The field line density immediately outside the cylinder is therefore zero.
to be constant in time, it must be in pressure equilibrium with its surroundings, so

\[ p_i + \frac{B^2}{8\pi} = p_o \quad (3.11) \]

where \( p_i \) is the gas pressure inside the cylinder and \( p_o \) is the gas pressure outside.

### 3.2 The MRI

It is the ability of magnetic field lines to exert tension that causes accretion disks in outburst to have such high levels of turbulence, and therefore accretion. One such way magnetic fields enhance angular momentum redistribution in disks is by the Magnetorotational instability\(^8\) (MRI), discovered by Steven Balbus and John Hawley in 1991\(^1\). Recall the hydrodynamic stability analysis of Keplerian disks explained earlier in this chapter. We found that a linearly displaced particle oscillates at the epicyclic frequency \( \kappa \) in equation 3.1. Imagine now that there is a vertical magnetic field \( B_z \) that threads the disk and we again perturb a fluid element. As a result of the flux freezing condition, the perturbed element will now experience a restoring force that we can, to first order, model as a spring with constant \( K \). Including this restoring force in the equations used to derive 3.1 we find that the epicyclic frequency has quite a different\(^3\) character:

\[ \kappa^2 = \frac{2K}{m} + R \frac{d\Omega^2}{dR}, \quad (3.12) \]

\(^8\)A realistic treatment depiction of the MRI here would be much too complex and cumbersome. I suspect it would add nothing to understanding this thesis and at worst obfuscate its general behavior, so it will not be included.
where $m$ is the mass of the fluid element. An instability will develop if $\kappa^2 < 0$ so a Keplerian rotation profile will become unstable when

$$K < \frac{3}{2} m \Omega^2.$$  

(3.13)

Therefore, the result of a sufficiently weak vertical magnetic within an accretion disk will be to destabilize perturbations in the disk plane!

![Diagram](image.png)

Figure 3.1: A simple depiction of the string analogy to the MRI. Two fluid elements placed at slightly different radii, labeled by $m_o$ and $m_i$ for the outer and inner radii respectively, will begin to separate due to the differential rotation profile of the disk. A weak magnetic field acts like a spring and creates a tension force that exchanges angular momentum from the inner to the outer mass, causing the inner mass to fall further in and the outer to do the opposite.

In keeping with this spring analogy, one is inclined to ask just what the other ‘end’ of the spring is attached to. Well, by invoking flux freezing as the reason for the restoring force, I made the implicit assumption that the inertia of surrounding fluid elements provided this force. Two neighboring fluid elements displaced by a small amount in radius will have slightly different angular velocities and so will begin to drift apart in their center of mass reference frame. As they drift, however, a magnetic restoring force
between the two will exert a torque on each. The torque on the inner element opposes its angular momentum while the torque on the outer reinforces its angular momentum. This torque therefore facilitates a transfer of angular momentum from the inner element to the outer. As this transfer takes place, the elements will find themselves out of equilibrium with their current orbits and so the inner (outer) element will fall (rise) to remain in orbital equilibrium at a radius of lower (higher) angular momentum. In this way, the MRI acts to exchange angular momentum between adjacent fluid elements and enhance the accretion rate.

### 3.3 Mean-Field Dynamos

While the MRI provides a great deal of insight into the nature of turbulence within accretion disks, it does not account for all of the turbulent stresses found in highly magnetized disks. Magnetic fields threading the disk can easily regenerate themselves through a process called a mean-field dynamo, the dynamics of which are principally captured by the induction equation, although in a slightly different form than in 3.4. To better capture the dynamics of the mean-field dynamo, we split the velocity and magnetic fields into their mean and fluctuating parts, respectively:

\[
\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}' \quad \text{and} \quad \mathbf{B} = \bar{\mathbf{B}} + \mathbf{B}'.
\] (3.14)

The overbar denotes an average that obeys Reynolds averaging rules:

\[
\bar{\mathbf{F}} + \mathbf{G} = \mathbf{F} + \bar{\mathbf{G}}, \quad \bar{\mathbf{F}}\mathbf{G} = \bar{\mathbf{F}}\mathbf{G}, \quad a\mathbf{F} = \bar{a}\mathbf{F}, \quad \text{and} \quad \mathbf{F}' = 0
\] (3.15)

where \(\mathbf{F}\) and \(\mathbf{G}\) are turbulent fields and \(a\) is a constant. Substituting the expansions 3.14 into the induction equation, we find that the mean and fluctuating magnetic fields...
evolve as
\[ \partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B} + \mathcal{E}), \tag{3.16} \]
and
\[ \partial_t \mathbf{B}' = \nabla \times (\mathbf{v} \times \mathbf{B}' + \mathbf{v}' \times \mathbf{B}) + \mathbf{G} \tag{3.17} \]
where \[\mathbf{G} = \nabla \times (\mathbf{v}' \times \mathbf{B}' - \mathcal{E})\] and \(\mathcal{E}\) is the mean electromotive force given by
\[ \mathcal{E} = \mathbf{v}' \times \mathbf{B}'. \tag{3.18} \]

The process by which a large-scale, coherent magnetic field forms is called a dynamo and is characterized by field growth through self-reinforcement. For accretion disks, \(\mathbf{v}\) is in theory\(^9\) very nearly zero in any region of dynamical interest and so advected large-scale fields play a very small role in the overall behavior of the dynamo. Surprisingly, the correlated turbulence captured in \(\mathcal{E}\) is the principle driver of the evolution of the mean field. Elucidating the origin of this correlated turbulence, however, is an incredibly difficult problem. While the evolution equation for \(\mathbf{B}'\) is already quite complicated, the same for \(\mathbf{v}'\) is much worse, since it is not only evolved by self-interaction with the mean-field \(\mathbf{B}\), but also with the turbulent field \(\mathbf{B}'\) (see Appendix ***** for a more detailed explanation). These feedback mechanisms are too complicated and unwieldy for us to use as interpretable closure relations, so in general we seek a much simpler expression for \(\mathcal{E}\) that will provide a sufficient closure relation for 3.16. This is usually done by assuming \(\mathcal{E}\) is a local quantity that only depends on \(\mathbf{B}', \mathbf{v}', \) and \(\mathbf{B}\) within a small radius. We additionally assume that the spatial and temporal variation in \(\mathbf{B}\) is sufficiently small that we can capture its behavior by its value and first spatial derivative where we evaluate \(\mathcal{E}\).

\(^9\)This is a different \(\mathcal{G}\) than used in equation 3.3 here it is called \(\mathcal{G}\) by historical convention and there it is simply a filler.
\(^{10}\)I will also show that this is true in practice.
Expanding, we have
\[ E_i = E_i^{(0)} + a_{ij}B_j + b_{ijk} \partial_k B_j \] (3.19)

where \( E_i^{(0)} \), \( a_{ij} \), and \( b_{ijk} \) are functionals of \( \mathbf{v}' \) and \( \mathbf{B}' \) but are totally independent of \( \mathbf{B} \).

Since \( \mathbf{v} = 0 \), the turbulence will be homogeneous and isotropic, meaning that all of the quantities in (3.19) must be isotropic as well. Consequently \( E_i^{(0)} = 0 \) (there are no isotropic vectors), \( a_{ij} = \alpha \delta_{ij} \) and \( b_{ijk} = \beta \epsilon_{ijk} \) where \( \alpha \) and \( \beta \) are averaged quantities (in the same sense as \( \mathbf{B} \)) depending on \( \mathbf{v}' \) and \( \mathbf{B}' \); \( \delta_{ij} \) and \( \epsilon_{ijk} \) are the Kronecker delta and Levi-Civita tensor, respectively. With these modifications, the closure relation is now
\[ \mathcal{E} = \alpha \overline{\mathbf{B}} - \beta \nabla \times \overline{\mathbf{B}}. \] (3.20)

The \( \alpha \) and \( \beta \) here are not to be confused with the entirely distinct yet identically-named \( \alpha \) from the alpha prescription (equation 1.4) and the plasma \( \beta \) (equation 3.9).

The calculations for \( \alpha \) and \( \beta \) are incredibly complicated and will not be detailed in this thesis. Instead I refer the reader to [10] for a more detailed analysis. After quite an intense calculation, however, \( \alpha \) and \( \beta \) show some remarkably simple relationships with \( \mathbf{v}' \) and \( \mathbf{B}' \).

For hydromagnetic turbulence, \( \alpha \) is composed of a kinetic and magnetic part [11]
\[ \alpha = \alpha_{\text{kin}} + \alpha_{\text{mag}} = -\frac{1}{3} [\mathbf{v}' \cdot (\nabla \times \mathbf{v}')] \tau_c + \frac{1}{3} [\nabla' \cdot (\nabla \times \mathbf{v}' A)] \tau_c, \] (3.21)

where \( \mathbf{v}'_A = \mathbf{B}' / \sqrt{4\pi \rho} \) is the fluctuating Alfvén velocity, and
\[ \beta = \frac{1}{3} (\mathbf{v}')^2 \tau_c. \] (3.22)

\( \tau_c \) is the turbulent correlation time, which is the same for all quantities**** and are defined by correlation integrals found in the appendix.
Chapter 4

The Simulations of Hirose et. al.

2014

The ideal MHD equations presented in section 3.1 are highly nonlinear and therefore require a numerical solution. We have no way to solve these equations to infinite precision\(^1\), so in general we choose a volume \(V\) in which to simulate their evolution and discretize the space. In three dimensions, this amounts to parsing \(V\) into voxels, or small volume elements for which it is assumed that certain values \((p, \rho)\) are constant within its volume or across its faces (e.g. \(v\) and \(B\)). Given the values of every relevant quantity for every voxel or voxel face in the grid at some time \(t\), we can solve for the value at some time \(t + \Delta t\), where \(\Delta t\) is a small time step bounded by the fastest physics that needs to be captured.

\(^1\)To do so would be unphysical anyway, since microscopic effects would quickly become important and vastly change the dynamics of the plasma
4.1 The Shearing Box Formalism and Full Equations

Global simulations of accretion disks are incredibly computationally expensive, so many alternatives to reduce this cost have been introduced. The alternative considered in this thesis, from [6], use a popular and well-documented[12] approach called the shearing box. The basic idea is as follows. Pick some fiducial radius $R_0$ with $R_0 \gg H$ so that the thin disk approximation is valid. Linearly expand all radial quantities about $R_0$ and define a local, co-rotating coordinate system $(x, y, z)$, corresponding to the radial, azimuthal, and vertical directions respectively. The radial extent of the box will be $(R_0 - L_x/2, R_0 + L_x)$, where $L_x \ll R_0$. Define the azimuthal extent $L_y$ such that the curvature in the disk is negligible and the vertical extent to ensure that $L_z > H$. For the simulations under present consideration, $(L_x, L_y, L_z) = (0.5H, 2.0H, 4.0H)$ and the grid resolution is 32x64x256 voxels in each direction respectively.

The full simulation equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$ (4.1)

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{\kappa_R \rho}{c} \mathbf{F},$$ (4.2)

$$\frac{\partial e}{\partial t} + \nabla \cdot (e \mathbf{v}) = -\left(\nabla \cdot \mathbf{v}\right) p - (4\pi B(T) - cE)\kappa_P \rho,$$ (4.3)

$$\frac{\partial E}{\partial t} + \nabla \cdot (E \mathbf{v}) = -\nabla \mathbf{v} : \mathbf{p} + (4\pi B(T) - cE)\kappa_P - \nabla \cdot \mathbf{F},$$ (4.4)

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0,$$ (4.5)
where, for completeness, $\rho$ is the density, $v$ the velocity, $p$ the thermodynamic pressure, $B$ the magnetic field, $\kappa_R$ and $\kappa_P$ the Rosseland and Planck mean opacities, $F$ is the radiation energy flux, $c$ is the speed of light, $e$ is the gas internal energy, $B(T) = \sigma_B T^4/\pi$ the Planck function, and $\sigma_B$ the Stefan-Boltzmann constant. Flux-limited diffusion is used to approximate the radiative transfer and relate the energy flux $F$ to the radiation pressure $P$ and radiation energy density $E$ (see [6] and [13] for more details on the flux-limited diffusion approximation). As a result of the shearing box not being in an inertial frame of reference, we add the inertial forces $\rho(-2\Omega\hat{z} \times v + 3\Omega^2 x \hat{x} - \Omega^2 z \hat{z})$ corresponding to the Coriolis force, tidal force, and local force of gravity. We must additionally impose a shear velocity within the box to model the differential rotation provided by the Keplerian angular velocity profile. Expanding $\Omega = \sqrt{GM/R^3}$ about $R_0$ we approximate the shear velocity $-q\Omega x$, where $q$ is the shear parameter ($q = 3/2$ for a Keplerian profile) and $\Omega$ is a constant for the simulation determined by $R_0$ and $M$.

The boundary conditions are shearing-periodic, periodic, and outflow for the $x$, $y$, and $z$ boundaries, respectively. Mathematically, these boundary conditions in $x$ and $y$ for any fluid quantity - except for $v_y$ in the radial boundary condition - $f$ are [14]:

$$f(x, y, z, t) = f(x + L_x, y - q\Omega L_x t, z, t) \quad (x \text{ boundary}, \quad (4.6)$$

$$f(x, y, z, t) = f(x, y + l_y, z, t) \quad (y \text{ boundary}). \quad (4.7)$$

Rewriting the azimuthal velocity as

$$v_y(x, y, z, t) = -q\Omega x + \delta v_y(x, y, z, t), \quad (4.8)$$

we find the radial boundary condition for $\delta v_y$ is the same for all other fluid quantities (equation [4.6]). These equations are numerically solved using the ZEUS code [15].
Figure 4.1: The general idea of a shearing box: instead of a global disk simulation, simulate a small patch of the disk at some radius $R_0$. Note the axes definitions are different than ours: our $x$ and $y$ axes point opposite to the ones depicted. We moreover include vertical stratification in our simulations, where the shearing box depicted here is of the unstratified type, which are only valid near the midplane.

4.2 Equation of State

In order to determine the gas pressure from temperature and density, we use the equation of state (EOS)

$$p = \frac{\rho}{\mu m_{\text{amu}}} \sigma_B T,$$

where $\mu$ is the mean molecular weight (in amu) and $m_{\text{amu}}$ is the atomic mass unit. Although 4.9 is, in fact, the equation of state for an ideal gas, its behavior is far from ideal. For these simulations, a solar composition of elements - i.e. that the relative number densities of elements in the sun match what are found in the accretion disk - is assumed and the ionization fraction of each is computed from the Saha equation. Not
only does the ionization fraction of each element contribute to the mean molecular mass, it also has a drastic effect on the gas’s Planck and Rosseland mean opacities.

The nonideal treatment of the EOS provides a much more realistic depiction of the thermodynamics within an accretion disk and plays a fundamental role in the long convective epochs we see near the end of the dwarf nova outburst phase. If an ideal EOS is used in the simulations, convection is much more infrequent and tenuous. Because convection is such an efficient means of energy transport, the disk cools rapidly and the temperature gradient quickly becomes subadiabatic. Nonidealities, however, result in a drastic increase in opacity during convection and allows the superadiabatic gradient to be maintained for much longer.

4.3 Suppression of Field Reversals

Here I provide an overview of the phenomenon that prompted the work in this thesis. During the early stages of the dwarf nova outburst phase, energy transport is facilitated entirely by radiation and the disk is transparent enough that the temperature gradient is subadiabatic. During these, what I call radiative epochs, the mean azimuthal magnetic field $B_y$ quasi-periodically flips sign with a period of 11 orbits (see figure 4.2, panel 1). It is currently not known why the dynamo produces this pattern and has been an enigma to scientists in the field ever since shearing box simulations began to come into use. However, as the disk loses surface density, the temperature gradient approaches the adiabat. Once this is hit, convection locks in as the transport mechanism as a result of the instability described in the previous section, ushering in a convective epoch. As shown in figure 4.2 (panel 2), a curious thing happens during convective epochs: $B - y$ no longer oscillates. This change in behavior is rather striking because it indicates that

\footnote{For example, $\mu$ is halved when a gas of pure hydrogen goes from completely unionized to a completely ionized plasma.}
the hydrodynamic turbulence caused by convection alters the accretion disk dynamo in a fundamental, observable way. Understanding how this modification happens could elucidate the cause of the butterfly diagram itself and lead to a better understanding of the high stresses produced in highly magnetized accretion disks.

Figure 4.2: Plots of $\bar{B}_y(z,t)$ for a simulation dominated entirely by radiative thermal transport (top panel) and a simulation with a long convective epoch (orbits 40-80, among others) on the bottom panel. The quasi-periodic flips in magnetic field in the radiative simulation are the defining characteristics of the butterfly diagram. Note that the changes in field cease during convective epochs.
Chapter 5

The Theory of Quasi-Periodic Field Reversals

The material presented in this and the next chapter is what I consider to be the main result of this thesis, namely a theory of how quasi-periodic reversals occur in accretion disks and how the phenomenon is altered by convection. This largely comes in two parts. The first, presented in this chapter, uses a phenomenological approach to dynamo theory to draw observational conclusions about the dynamo. While powerful, this method has its drawbacks in that it is only applicable at larger spatial and temporal scales. Therefore the second part, presented in the next chapter, principally leverages a classical description of dynamo theory as a means to stitch many of the observational conclusions together.

5.1 Introduction to Methodology

The symmetry and boundary conditions of the shearing box allow the induction equation to be highly simplified. Assuming the thin disk approximation is valid and that $R_0 \gg R_*$ (both of which are satisfied by the simulations under present analysis), the
simulated region is essentially homogeneous in the \( x \) and \( y \) directions and stratified only in \( z \). We leverage this symmetry for a simpler analytical picture by performing spatial averages over fluid quantities in the \( xy \) plane and denote the result with an overbar. For a fluid quantity \( f \), the horizontal average is defined as

\[
\bar{f}(z, t) = \frac{1}{L_x L_y} \int_{-L_x/2}^{L_x/2} \int_{-L_y/2}^{L_y/2} f(x, y, z, t) \, dx \, dy.
\] (5.1)

Correspondingly, we define the fluctuations in \( f \), \( f' \), by the equation \( f = \bar{f} + f' \). Applying this horizontal averaging technique to the induction equation 3.4, we find that many of the terms drop out in the averaging process as a result of the boundary conditions (equations 4.6 and 4.7) and we are left with

\[
\partial_t \bar{B}_x = -\partial_z (v_z \bar{B}_x - v_x \bar{B}_z),
\] (5.2)

\[
\partial_t \bar{B}_y = \partial_z (\delta v_y \bar{B}_z - v_z \bar{B}_y) - q \Omega \bar{B}_x,
\] (5.3)

\[
\partial_t \bar{B}_z = 0.
\] (5.4)

Note that \( \bar{B}_z \) has no horizontally-averaged sources - this does not mean that \( B'_z(x, y, z, t) = 0 \). The third source term for \( \bar{B}_y = -q \Omega \bar{B}_x \) is called shear and is due to the imposed velocity shear within the simulation. Physically, shear is the stretching of \( B_z \) into \( B_y \) from the differential rotation profile. The horizontally-averaged quantities \( \bar{v}_i \bar{B}_j \), which I will generally refer to as emfs, can be expanded into a more intuitive form. Expanding the velocity and magnetic fields into their horizontally-averaged and fluctuating parts, \( \bar{v}_i \bar{B}_j = (\bar{v}_i + v'_i)(\bar{B}_j + B'_j) = \bar{v}_i \bar{B}_j + v'_i \bar{B}_j' \), since the cross terms drop out due to Reynolds averaging rules. I will refer to emfs of the form \( \bar{v}_i \bar{B}_j \) as advective emfs and emfs of the form \( v'_i \bar{B}_j' \) as fluctuating or perturbative emfs. As a result, equations 5.2-5.4 take the

\[1\] There does not appear to be a standard name for emfs of this form in the literature, so I have invented this name for expediency.
form

\[ \partial_t B_x = -\partial_z (v_z B_x - \bar{v}_z \bar{B}_z) - \partial_z (v'_z B'_x - \bar{v}'_z \bar{B}'_z), \quad (5.5) \]

\[ \partial_t B_y = \partial_z (\delta v_y B_z - \bar{v}_z \bar{B}_y) + \partial_z (\delta v'_y B'_z - \bar{v}'_z \bar{B}'_y) - q \Omega \bar{B}_x, \quad (5.6) \]

\[ \partial_t B_z = 0. \quad (5.7) \]

Equations 5.5-5.7 should be thought of as the expansion of \( \partial_t \bar{B}_i \) in terms of its various components driven by specific physical mechanisms (emfs). For any \( \bar{B}_i \), if \( \partial_t \bar{B}_i \) shares the same sign, the magnitude of \( \bar{B}_i \) will be increasing; the opposite is true for opposite signs. Given this is true for the net change in field, it will be true for all of its components individually as well. So, by breaking apart the net \( \partial_t \bar{B}_i \) in terms of its components, we can understand how each of the individual physical sources for \( \bar{B}_i \) affect it and better understand its behavior.\(^2\)

### 5.2 Radiative Results

Starting with a simulation in the early stages of the outburst state in which the temperature gradient is well below the adiabat, I plot four separate manifestations of \( (\partial_t \bar{B}_y)_i \text{sign}(\bar{B}_y) \), shown in figure 5.1. The subscript denotes the panel (1-4) within figure 5.1 to which I am referring. These are: \( (\partial_t \bar{B}_y)_1 = \partial_t \bar{B}_y \), calculated by taking an explicit time derivative of \( \bar{B}_y \); \( (\partial_t \bar{B}_y)_2 = \partial_z (\delta v_y \bar{B}_z - \bar{v}_z \bar{B}_y) \), the effect of advective emfs; \( (\partial_t \bar{B}_y)_3 = \partial_z (\delta v'_y B'_z - \bar{v}'_z \bar{B}'_y) \), the effect of fluctuating emfs; and \( (\partial_t \bar{B}_y)_4 = -q \Omega \bar{B}_x \), the effect of shear on \( \bar{B}_y \).

First, let me state the obvious: the advective emfs only play a small role far away from the midplane where fast, coherent advection is possible as a result of the relatively

\(^2\)A special thanks to Matt Coleman for the original idea for these plots.
Figure 5.1: All plots here are of a simulation in which all heat transport is facilitated by radiation. Panels 1-4 here are: (1) the explicit time derivative of $B_y$, (2) advective emfs, (3) fluctuating emfs, and (4) shear - all multiplied by the sign of $B_y$ to show the effect that each has on the mean field. Advective emfs have very little effect on the evolution of the mean azimuthal field. Fluctuating emfs and shear are the principle contributors, working in general to decrease and increase the magnitude of the mean field, respectively.
low density and high magnetic forces present. Secondly, the incredibly noisy behavior of \((\partial_t B_y)_1\) is the result of two relatively coherent contributions to \(B_y\) - the fluctuating emfs and shear - which in general work to reduce and reinforce the field respectively.

### 5.2.1 The Source of Reversals

Given that the fluctuating emfs in panel 3 of figure 5.1 for the most part correspond to a \((\partial_t B_y)\) contribution with the opposite sign of \(B_y\), it is a valid question if they themselves are the source of field reversals. Plotting the contribution of each emf - \(\partial_z(\delta v'_y B'_z)\) and \(\partial_z(-v'_y B'_z)\) - separately in figure REF, we find that the first term dominates for \(|z| > 0.5H\) and is highly inactive toward the midplane, while the second completely dominates all the behavior below \(z = 0.5H\) and provides a contribution of similar magnitude to the first for \(0.5H < |z| < 1.0H\). The black lines on figures 5.1 and 5.2 denote contours where \(B_y = 0\) and signify a field reversal has taken place. Nearly all field reversals originate within half a scale height of the midplane, so if either term were to be responsible for a reversal, it would be the second. So, then, what is the physical origin of the emf? I propose that it must be the result of magnetic buoyancy created by shear.

**Magnetic Buoyancy**

Consider a magnetic flux tube of inner field strength \(B\) in an otherwise unmagnetized environment where there is a vertical force of gravity in the \(-\hat{z}\) direction. The flux tube will be in pressure equilibrium with its surroundings such that \(p_i + B^2/8\pi = p_e\) where \(p_i\) and \(p_e\) are the gas pressures inside and outside the tube. The high efficiency of radiation transport within the disk means that the flux tube will be in temperature equilibrium with its surroundings and we can assume that thermodynamic processes are isothermal. Then, \(\partial_i/p_e = \rho_i/\rho_e\), and since \(p_i < p_e\) as a result of the magnetic pressure inside the tube, \(\rho_i < \rho_e\). Thus the flux tube is buoyant and it will rise opposite the direction of
Figure 5.2: The contribution of fluctuating emfs 1 and 2 to $B_y$ are plotted, as a function of $z$ in scale heights and $t$ in orbits, in the first and second panel respectively. The first emf dominates for $|z| > 0.5H$, while the second mostly dominates for $|z| < 0.5H$.

5.2.2 Observational Evidence

Isothermal magnetic buoyancy matches extremely well with what we observe in the (radiation-dominated) simulations. For an isothermal expansion, $\partial P/\partial \rho = p/\rho$ and so $\partial \ln(p)/\partial \ln(\rho) = 1$, which is confirmed by the plot of $\delta p/p$ vs. $\delta \rho/\rho$ in figure 5.3 for $|z| < 0.25H$. Here and for the rest of the thesis,

$$\frac{\delta f}{f} = \frac{f - \bar{f}}{\bar{f}}$$  \hspace{1cm} (5.8)
for any fluid quantity \( f \). \( \bar{f} \) denotes the usual horizontal average and so \( \delta f / f \sim \partial \ln(f) \). In the same region near the midplane, figure 5.4 shows that perturbations in magnetic fields are correlated with lower than average densities (relative to their horizontal surroundings) and are therefore buoyant.

Figure 5.3: Cross correlation of \( \delta p/p \) vs. \( \delta \rho/\rho \) plotted for the Radiative simulation under analysis. The slope of 1 indicates that the thermodynamic processes are isothermal.

Buoyant flux tubes of the same sign as the mean field will produce a \( v_z' B_y' \) such that \( \text{sign}(v_z' B_y') = \text{sign}(z) \text{sign}(B_y) \). Given that this term works to suppress \( B_y \), \( \text{sign}(\partial_z - v_z' B_y') = -\text{sign}(B_y) \). Solving these equations, we find that \( v_z' B_y' \) must be increasing in magnitude away from the midplane in order to maintain the correct symmetry after - and it does. Figure 5.5 is a plot of the absolute value of \( v_z' B_y' \) as a function of \( z \), averaged over orbits 20-60. Although it is difficult to say why this gradient increases, it is most likely due to the fact that density begins to rarefy rapidly away from the midplane, resulting in the flux tubes meeting less drag resistance. There are many other confounding factors, however,
Figure 5.4: A plot of magnetic perturbations vs. density perturbations for the radiative simulation. The inverse correlation here suggests that high magnetic fields are correlated with regions of lower-than-average density and thus must be buoyant.

Discussion and Conclusions

Given the physical origin of \(-\partial_z v'_y B'y\) in magnetic buoyancy, this emf clearly cannot be a source of reversals. The emf is a reaction to a strong field, such that the dissipation rate is proportional (more or less) to the strength of the existing field. Mathematically, this can be modeled as a first order ODE, causing the field to decay in time. Simply, there is no reason to expect any oscillatory behavior here. \(\partial_z \delta v'_y B'_z\) should act in the same way. We do not observe a single reversal that originates from a region where this emf dominates, so it must act as a dissipation mechanism as well\(^3\). When the rate of

\(^3\)The origin of this saturation still remains a mystery; one possible mechanism is a Parker instability in which a finite \(k_y\) develops. Given density stratification, this term could provide the necessary correlations in \(\delta v'_y B'_z\) as well as the correct gradient symmetries.
The Theory of Quasi-Periodic Field Reversals

Chapter 5

Figure 5.5: Plot of $v'_{x}B'_{y}$ for the Radiative simulation, averaged over orbits 20-60. The magnitude of the emf increases away from the midplane symmetrically, so the symmetry in the gradient will be as desired.

Although magnetic buoyancy cannot be the source of a field reversal, it is likely the cause of the propagation of field reversals from the midplane outward to the edges of the disk. Consider a field reversal that has just started near the midplane, such that it is localized to a small area around the point at which the reversal began. Shear will produce buoyant flux tubes sharing the sign of the immediate mean field. As these tubes begin to rise, they will encounter the boundary between the ‘old’ field and the and the
mean field with which they share a sign. As the tubes hit this boundary, they will begin
to annihilate with the existing field at that boundary, expanding the area occupied by
the field reversal as they do. Buoyancy, therefore, etches out the existing field and makes
room for the new field. This may be why the buoyant rise time of magnetic flux tubes is
on the order of the butterfly diagram period.

5.3 Dynamo Modification by Convection

Consider figures 5.6 and 5.7 in which I have plotted all of the same quantities as
figure 5.1 and 5.2 except for a simulation with a long convective epoch between orbits
40 and 80. While the story is largely the same - advective emfs have very little effect
and the two fluctuating emfs affect $B^y$ away from and at the midplane, respectively - this
where the similarities end. $\partial_z(-v_z^zB^y)$, which previously functioned solely as a dissipation
mechanism for $B_y$, now functions as a source for $|z| < 0.25H$. Shear falls in to balance
the new source as a dissipation/reversal mechanism. In the coming paragraphs, I will
propose a reason (i.e. a physical mechanism) for why $\partial_z(-v_z^zB_y^y)$ switches sign and how
it suppresses field reversals itself; however, I will leave the discussion of its relationship
with shear to the next section.

Sign Flip of the EMF

The sign change in $\partial_z(-v_z^zB_y^y)$ near the midplane ($|z| < 0.25H$) during convection is
due to a sign change in $v_z^zB_y^y$ itself. The emf’s absolute value increases away from $z = 0$
just as before, (not shown, but checked), so the gradient cannot be responsible. It is
therefore either a sign change in the correlated velocity or magnetic field that causes
this behavior. Picking this behavior out of the disk turbulence, however, will be nearly
impossible as there is such a wide distribution of both within the disk. Luckily, though,
there is one thermodynamic quantity that we can leverage: temperature. The disk is so
Figure 5.6: All plots here are of a simulation in which there is a long convective epoch during orbits 40-80. Panels 1-4 here are: (1) the explicit time derivative of $\overline{B}_y$, (2) advective emfs, (3) fluctuating emfs, and (4) shear - all multiplied by the sign of $\overline{B}_y$ to show the effect that each has on the mean field. Again, advective emfs have very little effect on the evolution of the mean azimuthal field. Fluctuating emfs and shear are the principle contributors; however, their behavior has changed toward the midplane. Fluctuating EMFs no longer solely work to dissipate the field and visa-vera with shear: near the midplane we find that the fluctuating EMFs reinforce the mean field while shear works to dissipate it.
Figure 5.7: The contribution of fluctuating emfs 1 and 2 to $B_y$ are plotted for the convective simulation, as a function of $z$ in scale heights and $t$ in orbits, in the first and second panel respectively. There is still clearly a separation between the regions in which each emf is dynamically important. EMF 1 certainly dominates for the magnetically-dominated regions both during the radiative and convective epochs of the simulation. The second EMF clearly dominates for all hydrodynamically-dominated regions near the midplane; moreover, it appears to switch roles during convection from a dissipater to a reinforcer of the field.
opaque during convection that radiative heat transfer is essentially negligible and so the thermodynamic evolution of any structure as it moves throughout the disk (assuming it’s not destroyed) will be nearly adiabatic. With this in mind, I plot the cross correlation of $\delta T/T$ vs. $\overline{B_y}$ in figure 5.8. For orbits 50-60 (the domain over which the data for the figure is averaged), we find that there is a correlation between high magnetic fields of the same sign as the mean field of the inner disk and low temperature perturbations. Given the relatively low correlation of magnetic pressure perturbations $\delta P_{mag}/P_{mag}$ with temperature perturbations $\delta T/T$ in figure 5.9, it appears that the correlation observed in figure 5.8 cannot be caused by isothermal expansion from a buildup of magnetic field.

Figure 5.8: A plot of perturbations in temperature versus the value of $\overline{B_y}$ for orbits 50-60 of the convective simulation. The highest values of magnetic field are correlated with the coldest temperatures.

The origin of these fields, therefore, must be from somewhere relatively cold. This leaves us with one possibility: the fields were advected from outward in. Although the exact mechanism by which this happens is not well understood, it is most likely the
Figure 5.9: A plot of temperature perturbations versus perturbations in magnetic pressure for orbits 50-60 of the convective simulation. This correlation is mostly symmetric, so it is likely that adiabatic expansion under magnetic pressure is not creating the cold temperatures correlated with high $B_y$ in figure 5.8.

interaction between the extent of the convective turbulence (at roughly $z = \pm 0.25H$) and regions of high field intensity. Buoyant cells interacting with the strong field in this region could cause the field to undulate and fall into the convective downdrafts, getting sucked in with the strong ($\beta > 1$) hydrodynamical forces present. It is most likely, therefore, that the sign flip in the emf itself is a result of two phenomena: first, the downward (toward the midplane) hydrodynamic advection of magnetic field from the magnetically-dominated - and thus unaffected by convection - regions away from the midplane; and second, the alteration of magnetically-buoyant flux tubes from hydrodynamical turbulence, which is evident from the distribution of magnetic pressure and density perturbations presented in figure ****, which is highly altered from the cross correlation in the radiative simulation (figure 5.4).

Field Reversal Suppression
The mechanism by which these centrally-advected flux tubes suppress reversals is quite simple: they will either contribute to the strength of the field (if the two are of the same sign) or decrease the value of the mean field (in the case they are the opposite sign). Therefore, any mean field of the opposite sign as the advected field (which could begin to start a global reversal) will quickly vanish. Moreover, since magnetic buoyancy appears - at least for the most part - to be destroyed and/or modified by the strong hydrodynamic turbulence from convection, it is possible that a mean-field reversal could not effectively etch its way through the vertical extent of the accretion disk to impose a global reversal and is instead dissipated by the central advection of field from further away.
5.4 Feedback Mechanism

During the radiative simulations in which the motion of $B_y'$ was buoyant, shear nearly always worked to reinforce the mean field. There was no reason to assume any causal relationship between these two phenomena since it appeared to always work this way. However, now that we have a symmetric case in which sinking motions of $B_y'$ correspond to shear acting as a dissipation mechanism, we have reason to believe there is causality between the vertical direction of motion of $B_y'$ and the effect shear has on $B_y$. More specifically, this means that there must be a correspondence between $v_z' B_x'$ and one of the sources for $B_x$ - the most likely candidate being $\partial_z (v_z' B_y')$. If we assume this correspondence to zeroth order is simply that $B_x'$ and $B_y'$ lie in some fixed relation proportion to each other, a large enough $B_y'$ through buoyancy, could produce a reversal in $B_z$ that would in turn drive a reversal in $B_y$ and produce buoyant flux tubes, and on and on. This cycle could explain the emergence of the butterfly diagram.

Consider a spatially coherent, positive, mean radial field $B_x = c_x > 0$, that is instantaneously saturated so that $c_x$ is a constant. Through shear, an azimuthal magnetic field $B_y < 0$ will grow in magnitude linearly in time: $B_y(t) = -q \Omega c_x t$. This flux tube will start to become more and more buoyant as its magnetic pressure increases so will start to rise. This rising structure does not consist of purely $B_y'$ - it carries with it the original $B_x = c_x$, which hasn’t changed in magnitude since it has been assumed to be in equilibrium. However, as a result of the vertical rise of the tube, $B_x$ will begin to dissipate through the emf $-\partial_z (v_z' B_x')$ and so the tube will accumulate $B_y$ at a slower rate. The stability of this process will be dependent on the initial value of $c_x$. If $c_x$ is small, then the accumulation in $B_y$ will be small enough that the buoyancy of the tube never dominates the induction equation for $B_z$, resulting in a stable flux tube in which $\partial_t B_x$ is never less than zero. If, however $c_x$ is large, $B_y$ could accumulate so rapidly that the buoyant velocity forces the
term $-\partial_z (v'_z B'_z) \ll 0$ and results in a reversal in $B_x$ that causes a reversal in $B_y$ from shear, resulting in a field reversal within the disk$^4$.

**Two Dynamos**

The conclusion from the last two paragraphs, if true, is rather remarkable: the dissipation mechanism for the (relatively) coherent field $B_y$ plays a fundamental role in the smaller-scale dynamics of $B_x$. Essentially, since $\beta > 1$ ($\beta$ is the ratio of gas to magnetic pressure: equation 3.9), magnetic pressure could act on the background gas pressure to create buoyancy, which at high enough levels causes $B_x$ to flip sign and instigate a global field reversal. This implies that the dissipation mechanism $\partial_z (\delta v'_y B'_z)$ is actually representative of a dynamo with distinct characteristics, highly different than those found in the buoyant dynamo (by which I mean the dynamo discussed in the last two paragraphs). In figure ****, I have plotted the fluctuating emfs 1 and 2 multiplied by the sign of $B_y$, now with the contours on the plot corresponding to $\beta = 1$. The effect of the second fluctuating emf, which I will from now on refer to as the buoyant emf, on $B_y$ is almost completely characterized by the condition that $\beta > 1$. A similar statement can be said about the first fluctuating emf$^5$ and the condition $\beta < 1$, although it’s not quite as clear-cut. A better condition may be $\beta \leq 1$, since the regions in which it has a significant contribution while $\beta > 1$ correlate with regions of intense $B_y$ (this lends even more support to the idea that the field reduction is magnetic in nature and specifically dependent on the strength of $B_y$).

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$^4$This conclusion still needs to be hashed out in full detail to make sure that it is logically consistent.

$^5$I’m tempted to call this the Parker emf since it is almost certainly dissipation from a Parker instability, but will wait until the dissipation mechanism is fully understood to do so.
Chapter 6

Dynamo Theory: The Search for Closure

In light of the results of the previous chapter, this chapter is of a secondary importance to the previous. However, some possibly pertinent conclusions were made, so I will go ahead with presenting the chapter but in a much more brief way.

In section 3.3, the second-order correlation approximation (SOCA) was used to derive closure relations for $\mathcal{E}$ of the form:

$$
\mathcal{E} = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B}, \quad (6.1)
$$

where

$$
\alpha = \alpha_{kin} + \alpha_{mag} = -\frac{1}{3} \left[ \mathbf{v}' \cdot (\nabla \times \mathbf{v}') \right] \tau_c + \frac{1}{3} \left[ \mathbf{v}'_A \cdot (\nabla \times \mathbf{v}'_A) \right] \tau_c, \quad (6.2)
$$

$$
\left( \mathbf{v}'_A = \mathbf{B}' / \sqrt{4 \pi \rho} \right) \text{ and } \beta = \frac{1}{3} \left[ \mathbf{v}' \right]^2 \tau_c. \quad (6.3)
$$

After numerical reconstruction of these quantities from the simulations and calculat-
The fact that $\alpha$ plays little role in accretion disk turbulence is most likely due to the catastrophic quenching effect. $\alpha$ in equation 6.2 is essentially a balance between two different helicities: kinematic and magnetic. At the extremely high Reynolds numbers at which these simulations run, it has been suggested[?] that any influence to $\alpha$ from $\alpha_{\text{kin}}$ is immediately quenched by $\alpha_{\text{mag}}$ as a result of the symmetries of the shearing box and the fact that there is no source of helicity injection into the shearing box to excite a strong $\alpha$ term.
Figure 6.3: Here I plot $\eta_{xx}\partial_{zz} \bar{B}_x \text{sign}(\bar{B}_x)$ for the convective simulation. Clearly the turbulent resistivity acts as we expect - to everywhere diffuse the field.

Figure 6.4: Plotted is the same function as above, but for the radiative simulation.

Turbulent resistivity captured in $\beta$, however, provides a realistic closure relation. After numerical calculations of $\beta$, computing its contribution to $\partial_t \bar{B}_x$, and multiplying by the sign of $\bar{B}_x$, we obtain plots 6.3 and 6.4. The turbulent resistivity acts exactly how we would expect it to: it takes the second-order gradient of the field and does nothing but dissipate it. Similar plots for $\bar{B}_y$ are much less coherent than these - but that is probably to be expected. $\bar{B}_y$ is sourced by shear, which would have a small correlation with the present value of $\bar{B}_y$ and even less so with its second-order gradient. That being said, the highly coherent effect the turbulent resistivity has on $\bar{B}_x$ is incredibly promising in terms of constructing a closure relation with $\mathcal{E}$. 
Bibliography


