Quantum Supersymmetry!

Kristen Flowers
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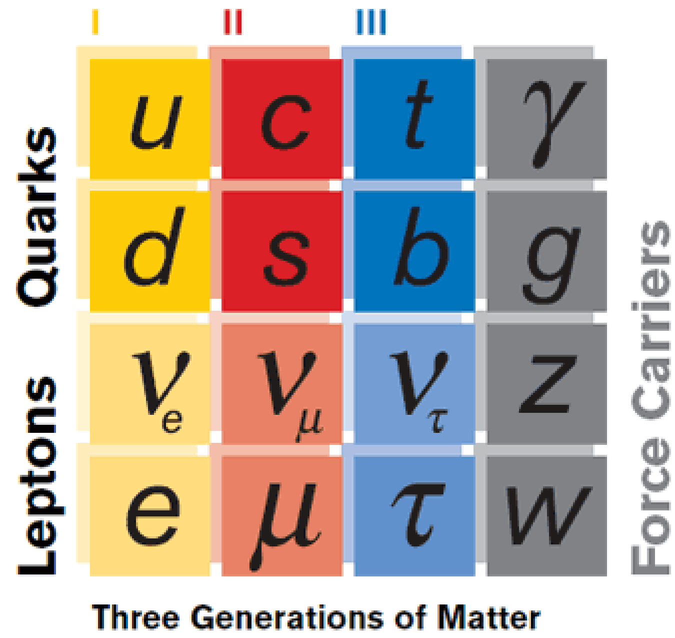
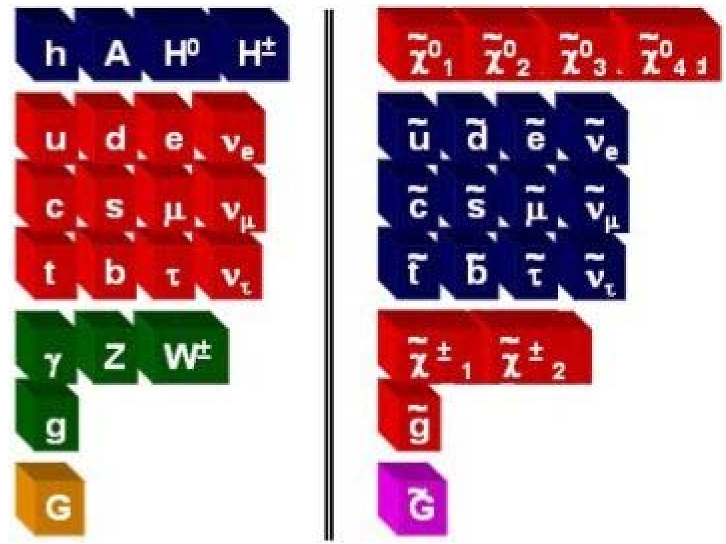
What's it all about?
Extensions to the Standard Model

Particle physicists suspect a deeper symmetry
 beneath the difference between bosons
 and fermions

Here, all known particles have a partner with
 a spin differing by ½ and all other quantum
 numbers equal
But since the symmetry is broken, their masses
 differ (and are slightly higher than
 energies explored so far)

Why its nice:
- Solves the mass hierarchy problem
- Provides framework for unifying particle
 physics and gravity
- Offers a dark matter candidate
- Keeps the Higgs from blowing up
- Other stuff, too





Minimal Supersymmetric Model (MSSM)

Standard Model + Two Higgs Doublets + SUSY Partners

Some models include R-parity conservation: $R=(-1)^{3(B-L)+2S}$

SUSY partners made in pairs, decay to the Lightest Stable Particle (LSP)

LSP would be electric+color neutral, only weakly interacting

Could be dark matter! (estimates ~ a few TeV... electroweak scale???)

Can't be too heavy, would appear in electroweak corrections

Some models include extra dimensions

In all, 124 parameters! (And you get 5 Higgses!)

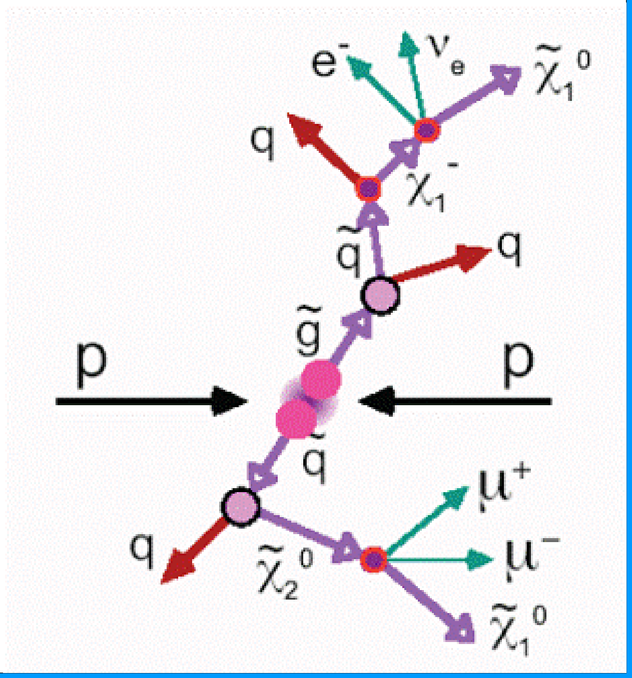
A particularly cool set of models:

Minimal Supergravity (mSUGRA)

Only the 18 SM parameters + 5 SUSY ones!

Experimentalists really like it...

.... plus theres gravity



Spacetime Symmetries!

Recall rotational symmetry and its operators...

Lorentz Group: rotations and boosts

in covariant form we see a rank-2 tensor generator M:

$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho}$

Two operators commute with all generators: S^2 and T^2 (Mandelstam variables!)

Poincare Group: rotations and boosts (M) + spacetime translations (P)

introduces two more commutation relations, 10 generators total:

$[P_\mu, P_\nu] = 0$

$[P_\mu, M_{\nu\rho}] = \eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu$

Commuting operators: P^2 (energy) and W^2 (spin)

Impossible to create a larger spacetime symmetry... unless anticommuting...

include anticommuting supercharges, a generalization of special relativity:

$[P_\mu, Q_a] = 0$ $\quad$ $[M_{\mu\nu}, Q_a] = -(1/2\ \sigma_{\mu\nu})_{ab}\ Q_b$

$\{Q_a, Q_b^\dagger\} = 2\ (\gamma^\mu)_{ab}\ P_\mu$ $\quad$ $\{Q_a, Q_b\} = -2\ (\gamma^\mu C)_{ab}\ P_\mu$ $\quad$ $\{Q_a^\dagger, Q_b^\dagger\} = 2\ (C^{-1}\gamma^\mu)_{ab} P_\mu$

Consequences of the SUSY Algebra

So, its another spacetime symmetry, with Q mixing fermions and bosons!

$\{Q,Q^\dagger\} = H$ $\{Q,Q\} = \{Q^\dagger,Q^\dagger\} = 0$ $[Q,H] = [Q^\dagger,H] = 0$

P^2 commutes with all generators, but no longer W^2

Particles in a supermultiplet have same mass, can have different spin

We restrict ourselves to the case of a single charge (because its physical!)

Representations can be labeled (m,j) for mass and spin with same number of states for integer and half integer spins

$P^2 = m^2>0$: each supermultiplet contains states $s=j\pm1/2$, and two $s=j$

j=0: 2 spin 0, 2 spin ½

j=1/2: 3 spin 1, 1 spin 0, 4 fermions

$P^2 = m^2=0$: states occur in pairs j, j-1/2 with CPT conjugates -j, -j+1/2

j=1/2: 2 fermionic states with helicity ±1/2, pair of spin 0 bosons

j=1: gauge bosons ±1, fermionic partners ±1/2

j=2: bosons ±2, fermions ±3/2 (supergravity!)

Recall a Familiar Example: The Harmonic Oscillator

$H_B = -\hbar^2/2m\ d^2/dx^2 + m\omega_B{}^2x^2/2$

We already know that it can be written in terms of raising/lowering operators:

$a = \sqrt{m\omega_B/2\hbar}\ (\ x + ip/m\omega_B\ )$ $a^\dagger = \sqrt{m\omega_B/2\hbar}\ (\ x - ip/m\omega_B\ )$ $[a,a^\dagger] = 1$

And we get the usual result with number operators:

$N_B = a^\dagger a$ $E_B = \omega_B(\ n_B + 1/2\ )$ and so, $H_B = \omega_B/2\ \{\ a^\dagger,\ a\ \}$

Now lets try something... plug in ladder operators under $\{b^\dagger,b\} = 1$

$H_F = \omega_F/2\ [\ b^\dagger,\ b\ ]$ therefore: $E_F = \omega_F(\ n_F - 1/2\ )$ where: $N_F = b^\dagger b$

$E_S = \omega_F(\ n_F + 1/2\ ) + \omega_B(\ n_B - 1/2\ ) = \omega(\ n_F + n_B\ )$ $\omega_F=\omega_B$ SUSY unbroken

On states $|\ n_B\ ,\ n_F\ \rangle$, it is obvious that ladder operators individually create/destroy each n seperately:

$a\ |\ n_B\ ,\ n_F\ \rangle = |\ n_B -1,\ n_F\ \rangle$ $a^\dagger\ |\ n_B\ ,\ n_F\ \rangle = |\ n_B +1,\ n_F\ \rangle$

$b\ |\ n_B\ ,\ n_F\ \rangle = |\ n_B\ ,\ n_F -1\ \rangle$ $b^\dagger\ |\ n_B\ ,\ n_F\ \rangle = |\ n_B\ ,\ n_F +1\ \rangle$

Connecting Our Simple Example to the Symmetry Group:

We know the established SUSY group requires: $H = \{Q,Q^\dagger\} = QQ^\dagger + Q^\dagger Q$

So from H_S in terms of the bosonic/fermionic operators

$$H_S = \omega (N_B + N_F) = \omega (b^\dagger b + a^\dagger a)$$

satisfied when: $Q = \sqrt{\omega}\, ab^\dagger$ and $Q^\dagger = \sqrt{\omega}\, a^\dagger b$

the other commutation relations follow, indicating a conserved quantity:

$\{Q,Q\} = \{Q^\dagger,Q^\dagger\} = 0$ $[Q,H] = [Q^\dagger,H] = 0$

Applied to our states:

$Q \,|\, n_B , n_F \rangle = \sqrt{\omega}\, |\, n_B -1, n_F +1 \rangle$ $(n_B , n_F \neq 0)$

$Q^\dagger \,|\, n_B , n_F \rangle = \sqrt{\omega}\, |\, n_B +1, n_F -1 \rangle$ $(n_F \neq 0)$

Cool, Q's convert between fermions and bosons without changing the energy!

(provided its unbroken, and $\omega_F=\omega_B$)

Lets look more at the relationship between H and Q...

Factoring the Harmonic Oscillator Hamiltonian:

Start with the harmonic oscillator, shifted so the ground state energy is zero:

$$H = -\hbar^2/2m\ d^2/dx^2 + m\omega^2x^2/2 - \hbar\omega/2$$

SUSY also suggests a way to factor our Hamiltonian into pieces, by representing the fermionic operators as Pauli matrices:

$$N_F = b^\dagger b = ½\,(\sigma_x + i\sigma_y)\ ½\,(\sigma_x - i\sigma_y)$$

$$H = \begin{vmatrix} \omega a^\dagger a & 0 \\ 0 & \omega a a^\dagger \end{vmatrix} = \begin{vmatrix} H_1 & 0 \\ 0 & H_2 \end{vmatrix} \quad \text{therefore, } Q^\dagger = \begin{vmatrix} 0 & \sqrt{\omega}\, a^\dagger \\ 0 & 0 \end{vmatrix} \quad Q = \begin{vmatrix} 0 & 0 \\ \sqrt{\omega}\, a & 0 \end{vmatrix}$$

So the operator converting between H_1 and H_2 is therefore:

$\sqrt{\omega}\, a = A$ $\quad \sqrt{\omega}\, a^\dagger = A^\dagger$ $\quad$ (applying A to H_n corresponds to Q on H_S)

$$H = \begin{vmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{vmatrix}$$ so H_S is factored into two pieces: $H_1 = A^\dagger A$ $\quad H_2 = AA^\dagger$

Now we can use this general way to factor all "supersymmetric" Hamiltonians!

Supersymmetric Partner Hamiltonians H_1, H_2

Consider the general case, with the ground state shifted to zero

$-\hbar^2/2m\ d^2/dx^2\ \Psi_0 + V(x)\ \Psi_0 = 0$ so, $V(x) = \hbar^2/2m\ \Psi_0''/\Psi_0$

Using the previous factorization:

$A = \hbar/\sqrt{2m}\ d/dx + W(x)$ $A^\dagger = -\hbar/\sqrt{2m}\ d/dx + W(x)$ so, $V = W - \hbar/\sqrt{2m}\ W'$

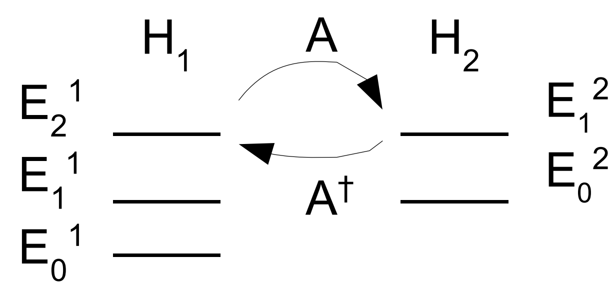
Relating the energy spectra and eigenvalue equations:

$H_1\Psi_n = A^\dagger A\Psi_n = E_n^1\Psi_n$ $H_2(A\Psi_n) = AA^\dagger A\Psi_n = E_n^1 A\Psi_n$ (likewise for H_2)

So we get a (nearly) doubly degenerate energy spectrum:

$E_0^1 = 0$ $E_n^2 = E_{n+1}^1$ $\Psi_n^2 = (E_{n+1}^2)^{-1/2} A\Psi_{n+1}^1$ $\Psi_{n+1}^1 = (E_n^2)^{-1/2} A^\dagger\Psi_n^2$

A ($A^\dagger$) converts an eigenfunction of H_1 (H_2) into an eigenfunction of H_2 (H_1) with the same energy, while creating (destroying) a node



Another Example of Supersymmetric Potentials

For example, start with the ground state of infinite square well: $V_1 = 0$ $(0<x<L)$

$\Psi_0^1 = \sqrt{2/L}\,\mathrm{Sin}(\pi x/L)$ $\qquad E_0 = \hbar^2\pi^2/2mL^2$

Subtract off ground state, we get:

$\Psi_n^1 = \sqrt{2/L}\,\mathrm{Sin}((n+1)\pi x/L)$ $\qquad E_n = \hbar^2\pi^2 n(n+2)/2mL^2$

So the corresponding superpotential is:

$W = -\hbar/\sqrt{2m}\,\Psi_0'/\Psi_0 = -\hbar/\sqrt{2m}\,\pi/L\,\mathrm{Cot}(\pi x/L)$

$V_2 = \hbar^2\pi^2/2mc^2\,(2\,\mathrm{Cosec}^2(\pi x/L) - 1)$

Wavefunctions of H_2 are found by applying A to H_1- same spectra but shifted!

$\Psi_0^2 = -2\sqrt{2/3L}\,\mathrm{Sin}^2(\pi x/L)$ $\qquad \Psi_1^2 = -2/\sqrt{L}\,\mathrm{Sin}(\pi x/L)\,\mathrm{Sin}(2\pi x/L)$ etc

Can also do for continuous spectra, like scattering

This technique allows us to easily solve for all kinds of potentials, a useful tool!

The Infinite Square Well and its Supersymmetric Partner Potential: