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Analysis of Quantum Circuits

Classical circuits have been described in detail in a number of good references such as The Art of Electronics by Horowitz & Hill. Passive circuits eventually just come down to solving Kirchoff Law equations that are a statement of

1. The sum of voltages around a loop must be zero. (Energy Conservation)

2. The sum of current into and out of a node must be zero. (Charge conservation)

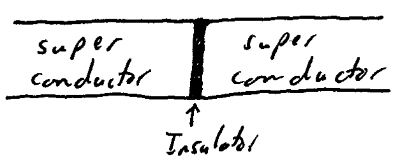
For LCR circuits there is a well known correspondence to harmonic oscillators.

However, in superconducting circuits, particularly those with quantum circuit elements like the Josephson junction, an explicitly Hamiltonian formulation of the circuit is needed to bring the Schrödinger formalism to bear.

Since an absolute necessity for dealing with quantum circuits is the treatment of Josephson junctions, We will begin by introducing variables that naturally describe them.

Josephson Junctions

A Josephson junction consists of a thin oxide or other insulating layer between two super conductors.



The dielectric layer is typically on the order of a few nanometers. This allows tunneling of cooper pairs through the dielectric layer without dissipation. It can be shown that the phase of the cooper pairs have different values on the two sides of the junction. A detailed description of this requires a second quantized formalism that is then used for second order perturbation or a scattering calculation. See

"Superconducting Qubits and the Physics of Josephson Junctions" by J. Martinis and K. Osborne

The important result of this calculation is the Josephson relations:

$$I_J = I_0 \sin\delta \qquad (1)$$

$$V = \frac{\Phi_0}{2\pi}\frac{d\delta}{dt} \qquad (2)$$

where I_J is the super current, V is the voltage across the junction, δ is the phase difference, I_0 is the critical current of the junction, and $\Phi_0 = h/2e$ is the flux quantum.

Differentiating (1) and using (2) to substitute in yields

$$\frac{dI_J}{dt} = (I_0 \cos\delta)\,\frac{2\pi}{\Phi_0}\,V$$

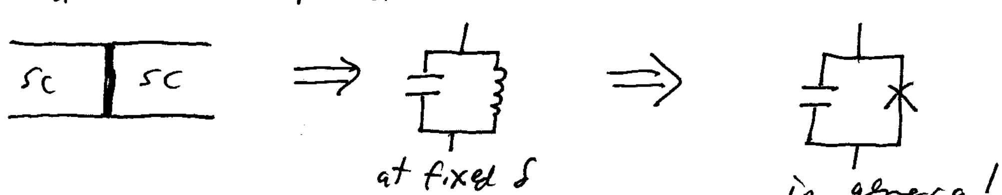
Recalling the equation describing a classical inductor is

$$V = L\frac{dI}{dt}$$

we can define an inductance associated with the junction as

$$L_J = \frac{\Phi_0}{2\pi I_0 \cos\delta} \qquad (3)$$

Since there is capacitance associated with the faces of the superconductors, we can naturally treat the junction as an LC oscillator:



But notice that by integrating (2) we get that the relevant variable is

$$\Phi = \frac{\Phi_0}{2\pi}\delta = \int_{-\infty}^{t} V(t)\,dt \qquad (4)$$

For symmetry we also introduce

$$Q = \int_{-\infty}^{t} I(t)\,dt \qquad (5)$$

where Φ is the flux through the circuit and Q is the charge having flown through the junction. Since the carriers are cooper pairs

$$Q_J = -2eN(t) \qquad (6)$$

A Hamiltonian Formalism

From the discussion of ~~Johs.~~ Josephson junctions it is clear that the convenient variables will be the flux and charge variables. In general, however we need a convenient way to relate adjacent elements in the circuit to each other. This will mean either Kirchoff's laws in Q and Φ or introducing node variables that already depend on the geometry of the circuit. In some sense this is related to, for example, choosing spherical coordinates over cartesian coordinates.

In order to prevent double counting, we introduce the idea of a spanning tree to organize any arbitrary passive circuit.

Start by defining a ground node. The actual location does not matter and corresponds to choosing an origin. For the ground node, the node flux is zero, i.e.

$$\phi_{GND} = 0 \qquad (7)$$

Next choose a set of circuit elements that provide exactly one path to ground for every active (non-ground) node. This set is the spanning tree and each element is a branch. The remaining branches are called closure branches and each one defines a loop by joining the two ends by the shortest path on the spanning tree. These loops will be "irreducible" in the sense that no loop will contain any branch more than once.

We can then define node fluxes as

$$\phi_n = \sum_{tree} S_{nb} \Phi_b \qquad (8)$$

where Φ_b is the branch flux defined in (4) and S_{nb} is 1 for branches with positive orientation, -1 for branches with reversed orientation, and 0 for branches not connecting node n to ground.

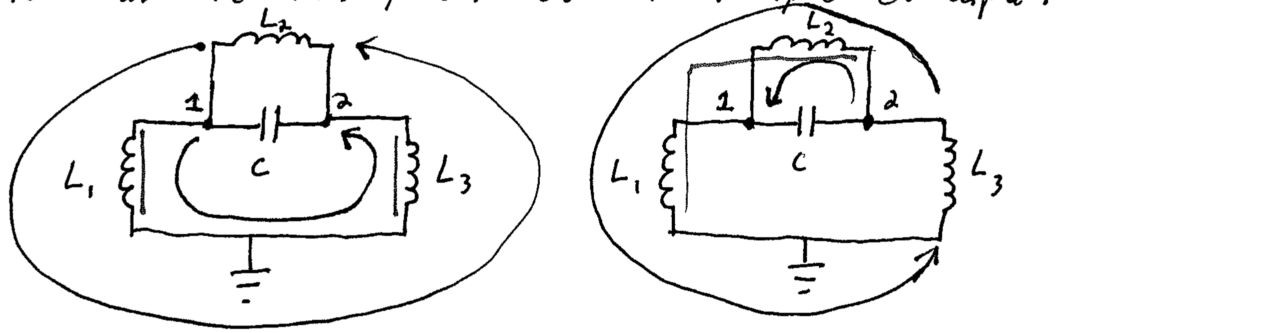
We can then relate branch and node fluxes by

$$\Phi_{b \in tree} = \phi_n - \phi_{n'} \qquad (9)$$

$$\Phi_{b \in closure} = \Phi_{loop} - (\phi_n - \phi_{n'}) \qquad (10)$$

Note that the spanning tree defines ϕ's so to get the branch fluxes of the closure branches we take the total flux through the loop defined by the closure branch and then subtract off the part from the spanning tree.

To illustrate this, let's do a simple example.



Above are two choices of spanning trees (red) with their accompanying irreducible loops (blue). Notice that the loop with L_2 and C does not appear on the left. Attempting to use all 3 loops with either Kirchoff's laws or the formalism being developed in the next section would yield dependent equations. As circuits become more complicated, eliminating unnecessary loops will be more important and harder to do without the spanning tree.

Note that the placement of ground is arbitrary. Also, the capacitor could have been taken on the spanning tree, but we will see shortly that the Φ_{loop} term in (10) has much more impact on inductors, so given the choice we take inductors on the spanning tree.

A circuit Lagrangian

Now that we have a reasonable set of generalized coordinates, we would like to write down a Lagrangian. Let's start by getting equations for the energy stored in an inductive or capacitive element. Since inductors and capacitors ~~are~~ have the same equations under $L \leftrightarrow C$, $I \leftrightarrow V$ (i.e. inductors are dual to capacitors), we need only solve one to get everything for the other. For inductors,

$$V = L\frac{dI}{dt} \qquad (11)$$

From the definition of the branch flux (H)

$$\begin{aligned}\Phi_b &= \int_{-\infty}^{t} V(t)\,dt \\ &= \int_{-\infty}^{t} L\frac{dI}{dt}\,dt \\ &= L\int_0^{I(t)} dI' \\ &= L\,I(t)\end{aligned}$$

$$\Rightarrow I(\Phi) = \frac{\Phi}{L} \qquad (12)$$

The energy stored in an inductor is

$$E_L = \int_{-\infty}^{t} L I \frac{dI}{dt}\, dt \quad (13) \text{ from Griffiths}$$

$$= \int_0^{I(t)} \Phi \, dI$$

$$= \int_0^{\Phi} \frac{\Phi'}{L}\, d\Phi$$

$$E_L = \frac{\Phi^2}{2L} \qquad (14)$$

Using the dual relationship for capacitors

$$E_C = \frac{Q^2}{2C} \qquad (15)$$

However these are only for linear elements, but (13) was completely general. Using (12) and (13) yields

$$E_L = \int_0^{\Phi} I(\Phi)\, d\Phi \qquad (16)$$

$$E_C = \int_0^{\Phi} V(Q)\, dQ \qquad (17)$$

These will work for Josephson junctions and crazy nonlinear capacitors if such a thing exists.

Since Φ is related to the generalized coordinates for our circuit, the node fluxes ϕ, we really want (15) in terms of Φ. Assuming a linear capacitor,

$$\Phi_C = \int_{-\infty}^{t} V(t)\, dt \qquad \text{from } (4)$$

$$= \int_{-\infty}^{t} \frac{Q}{C}\, dt$$

$$\Rightarrow \dot{\Phi}_C = \frac{Q}{C} \qquad (18)$$

We can then rewrite (15) as

$$E_C = \frac{1}{2} C \dot{\Phi}^2 \qquad (19)$$

At this point it is clear that capacitors correspond to kinetic energy in the ~~Hamiltonian~~ Lagrangian and inductors to potential energy. Using (9) and (10) we can write the Lagrangian.

$$\cancel{L = \frac{1}{2} C \Sigma}$$

$$L = \sum_{\text{tree}} \frac{1}{2} C_i \left(\frac{d}{dt}\left(\phi_i - \phi_{i'}\right)\right)^2 + \sum_{\text{closure}} \frac{1}{2} C_i \left(\frac{d}{dt}\left(\Phi_{\text{loop}} - (\phi_i - \phi_{i'})\right)\right)^2$$

$$- \left[\sum_{\text{tree}} \frac{1}{2L}\left(\phi_i - \phi_{i'}\right) + \sum_{\text{closure}} \frac{1}{2L}\left(-\phi_i + \phi_{i'} + \Phi_{\text{loop}}\right)\right)$$

Noting that Φ_{loop} is constant in time for constant flux through the circuit,

$$L = \sum \tfrac{1}{2} C_i (\dot{\phi}_i - \dot{\phi}_{i'})^2 - \left[\sum_{tree} \frac{1}{2L_j} (\phi_j - \phi_{j'})^2 + \sum_{closure} \frac{1}{2L_k} \left(\Phi_{loop} - (\phi_k - \phi_{k'}) \right)^2 \right] \quad (20)$$

We can get the cannonically conjugate node charges

$$q_i = \frac{\partial L}{\partial \dot{\phi}_i} \quad (21)$$

This turns out to be the total charge on the capacitors connected to a particular node. Since any node will have capacitance to ground from the parasitics of the circuit, this quantity is always defined.

We can get the Hamiltonian in the usual way

$$H = \sum_i \dot{\phi}_i q_i - L(\phi, \dot{\phi}, t) \quad (22)$$

At this point we can get the hamiltonian for any LC circuit. We define a spanning tree and then use equations (20), (21) and (22).

A few points are worth mentioning before moving on to do quantum mechanics. Notice that in (20) the $(\dot{\phi}_i - \dot{\phi}_{i'})^2$ term has no analogue in spring systems unles $\dot{\phi}_{i'}$ is zero. ~~For~~ The Φ_{loop} terms act as an offset for the potential which also typically does not appear in springs but cannot be eliminated from this formalism.

Since this is just classical mechanics and because every node requires additional capacitance to ground, I will forgo an example. An LC circuit is trivial and the next simplest circuit is the one on page 4 with two capacitors to ground, which is just a lot of algebra with no real interesting resut.

Quantum at Last

Now that we have a complete Hamiltonian formalism, all we need to do is introduce operators.

$$\phi \rightarrow \hat{\phi}$$

$$q \rightarrow \hat{q}$$

$$H \rightarrow \hat{H}$$

Since q and ϕ are cannonically conjugate

$$[\hat{\phi}, \hat{q}] = i\hbar \qquad (23)$$

In some sense the point of all this has been to develop a formalism to deal with Josephson junctions. And now we finally are in a position to do that.

The charge on the junction is simply the charge of Cooper pairs that have tunneled through. Define then a number operator for tunneled Cooper pairs.

$$\Rightarrow \quad \hat{Q} = -2e\hat{N} \qquad (24)$$

$$\hat{N} = \sum_N N |N\rangle\langle N| \qquad (25)$$

Recall also that

$$\hat{\Phi} = -\frac{\hbar}{2e}\hat{\delta} \qquad (26) \qquad \text{from } (4)$$

We note that $[\hat{Q}, \hat{\Phi}] = i\hbar$ if the branch is on the spanning tree and connected to ground. In this case,

$$[\hat{Q}, \hat{\Phi}] = [\hat{q}, \hat{\phi}] = i\hbar$$

But commutators are independent of coordinates system. Hence

$$[\hat{Q}, \hat{\Phi}] = i\hbar$$

$$[-2e\hat{N}, -\tfrac{\hbar}{2e}\hat{\delta}] = i\hbar \Rightarrow [\hat{N}, \hat{\delta}] = i \qquad (27)$$

So we can see that $\hat{N}$ and $\hat{\delta}$, which are the relevant operators for the Josephson junction, are really conjugate.

The Hamiltonian in the N basis will couple ~~too~~ adjacent N states together with the energy cost for tunneling a Cooper pair through the junction, viz.

$$H_J = -\frac{E_J}{2}\sum_{N=-\infty}^{\infty}\left[|N\rangle\langle N+1| + |N+1\rangle\langle N|\right] \qquad (28)$$

Of course the phase turns out to be the useful value since it is ~~conjugate~~ related to the flux which is the basis of our formalism.

We start by just writing down an $|N\rangle$ and inserting a completeness relation.

$$|N\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\delta\ |\delta\rangle\langle\delta|N\rangle \qquad (29)$$

Note that, δ is a "position" variable that lies on the interval $[0, 2\pi]$ and N is its roughly conjugate variable ~~with~~ and is hence the generator of δ translations, In otherwords, up to factors out front, δ and N are the same as ϕ and σ_z. consider

$$\hat{N}|N\rangle = N|N\rangle$$

$$\langle\delta|\hat{N}|N\rangle = N\langle\delta|N\rangle$$

$$-\frac{\partial}{\partial\delta}\langle\delta|N\rangle = N\langle\delta|N\rangle \qquad \text{from Sakurai 3.6.9}$$

$$\Rightarrow \langle\delta|N\rangle = e^{-iN\delta} \qquad (30)$$

Hence we can write

$$|N\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\delta\, e^{-iN\delta}|\delta\rangle \qquad (31)$$

$$|\delta\rangle = \sum_N e^{iN\delta}|N\rangle \qquad (32)$$

Consider then a term from (28)

$$|N\rangle\langle N+1| = \frac{1}{2\pi}\int_0^{2\pi} d\delta\, e^{-iN\delta}|\delta\rangle\frac{1}{2\pi}\int_0^{2\pi} d\delta'\, e^{+i(N+1)\delta'}\langle\delta'|$$

$$= \frac{1}{(2\pi)^2}\int_0^{2\pi} d\delta\int_0^{2\pi} d\delta'\, e^{-iN(\delta'-\delta)}\, e^{i\delta'}\,|\delta\rangle\langle\delta'|$$

Doing the integral over δ, this is nonzero only for $\delta' = \delta$ and $|N\rangle\langle N+1| \neq 0$

$$= \frac{1}{N}\frac{1}{2\pi}\int_0^{2\pi} d\delta'\, e^{i\delta'}|\delta'\rangle\langle\delta'|$$

The Hamiltonian (28) becomes in the δ basis

$$H_J = -\frac{E_J}{2} \sum_N \frac{1}{N} \frac{1}{2\pi} \int_0^{2\pi} d\delta \left(e^{i\delta} + e^{-i\delta}\right) |\delta\rangle\langle\delta|$$

$$= -\frac{E}{2} \frac{1}{2\pi} \int_0^{2\pi} d\delta \left(e^{i\delta} + e^{-i\delta}\right) |\delta\rangle\langle\delta|$$

We now define

$$e^{i\delta} = \frac{1}{2\pi} \int_0^{2\pi} d\delta\, e^{i\delta} |\delta\rangle\langle\delta|$$

Noting that

$$e^{i\delta}|\delta'\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\delta\, e^{i\delta} |\delta\rangle\, \delta_{dirac}(\delta - \delta')$$

$$= e^{i\delta'}|\delta'\rangle$$

$$\Rightarrow \boxed{H_J = -E_J \cos\delta} \qquad (33)$$

While interesting, we note that we never invoked the Josephson relations. Classically, or semiclassically since Josephson junctions are strictly quantum mechanical, using (16)

$$E_L = \int_0^{\Phi} I(\Phi)\, d\Phi$$

$$= \int_0^{\Phi} I_0 \sin\delta\, d\Phi \qquad \text{from} \quad (1)$$

$$= \int_0^{\delta} \frac{\hbar}{2e} I_0 \sin\delta\, d\delta$$

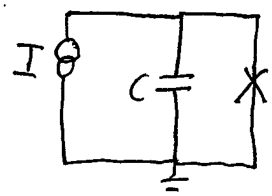
$$= -\frac{\hbar}{2e} I_0 \cos\delta$$

Evidently, $E_J = \frac{\hbar}{2e} I_0$ and we could have just made δ an operator. It's worth noting that since E_J, the tunneling energy, is related to the superconductors and the geometry of the junction, the critical current is also.

We can finally put this all together and solve a qubit!

Phase Qubits

The prototypical phase qubit looks like this



We can treat the current source as a really big inductor with big flux so that $\tilde{\Phi}_s / L_s = I$. Taking the junction as the spanning tree, from (20)

$$T = \frac{1}{2} C \dot{\phi}^2$$

$$V = -E_J \cos \frac{2e}{\hbar} \phi + \frac{(\tilde{\Phi}_s - \phi)^2}{2 L_s}$$

$$L = \frac{1}{2} C \dot{\phi}^2 + E_J \cos \frac{2e}{\hbar} \phi - \frac{(\tilde{\Phi}_s - \phi)^2}{2 L_s}$$

$$q = C \dot{\phi}$$

$$\Rightarrow H = \dot{\phi}(C\dot{\phi}) - \frac{1}{2} C \dot{\phi}^2 - E_J \cos \frac{2e}{\hbar} \phi + \frac{(\tilde{\Phi}_s - \phi)^2}{2 L_s}$$

$$H = \frac{q^2}{2C} - E_J \cos \frac{2e}{\hbar} \phi + \frac{(\tilde{\Phi}_s - \phi)^2}{2 L_s}$$

$$= \frac{q^2}{2C} - E_J \cos \delta + \frac{\tilde{\Phi}_s^2}{2L_s} - 2\frac{\tilde{\Phi}_s \phi}{2L_s} + \frac{\phi^2}{2L_s}$$

$$= \frac{q^2}{2C} - E_J \cos \delta + \frac{\tilde{\Phi} I}{2} - I\phi + \frac{\phi^2}{2L_s}$$

We now note that L_s is huge so the last term is negligible. The $\tilde{\Phi}$ term is a constant offset to the potential, so lets just set that to zero.

$$\Rightarrow H = \frac{q^2}{2C} - I\phi - E_J \cos \frac{2e}{\hbar} \phi$$

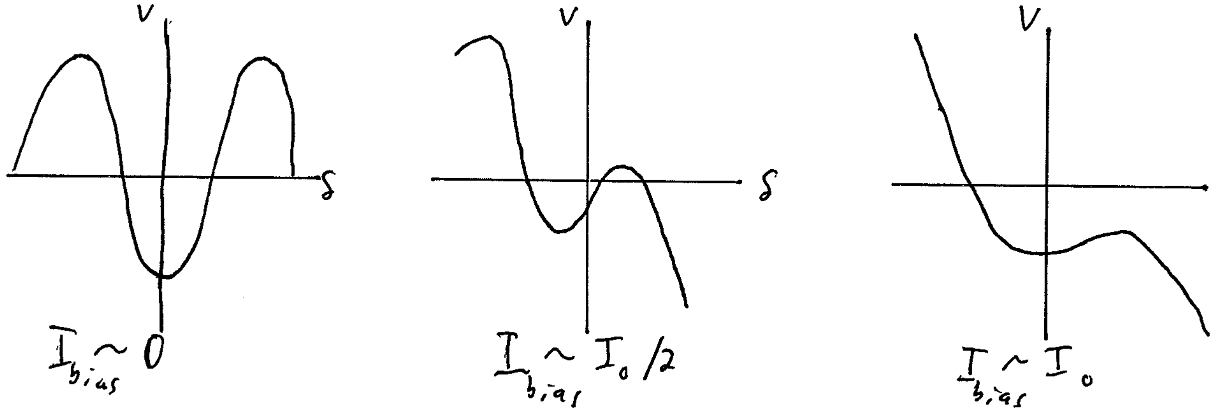
$$\cancel{H = \frac{q^2}{2C} - \frac{2e}{\hbar} I \delta - E_J}$$

$$\boxed{H = \frac{q^2}{2C} - I \Phi_0 \delta - I_0 \Phi_0 \cos \delta} \qquad (34)$$

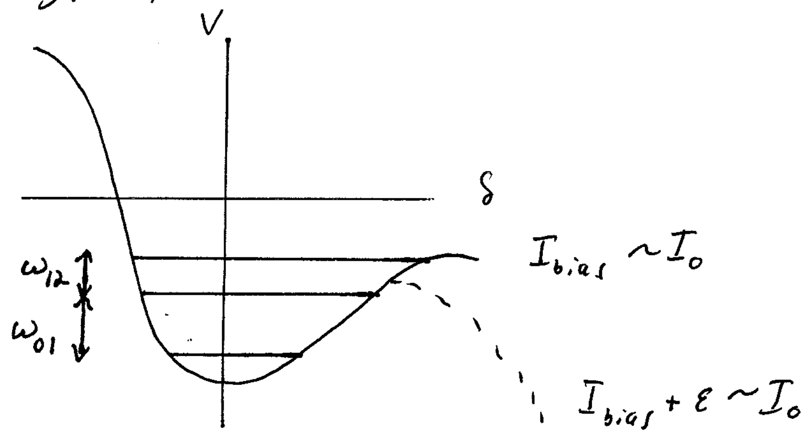
The interesting part of the Hamiltonian is the potential

$$V = -\Phi_0 \left[I_{bias}\delta + I_0 \cos\delta \right] \qquad (35)$$

Plotting this for several I_{bias} (the bias current),



Notice at higer bias current the well becomes shallower and more asymmetric. For $I_{bias} \sim I_0$, consider the energy diagram



Note that there are only a few modes in the well and their energy separation is different. This means that if we wanted to, we could excite transitions between the lowest modes without worrying about higher modes being excited. This is the most basic requirement for a qubit.

Furthermore, there is a built in measurement system. By further increasing I_{bias}, the well becomes shallow and the $|1\rangle$ state becomes likely to tunnel out of the well. The second Josephson relation

$$V = \frac{\Phi_0}{2\pi} \frac{d\delta}{dt} \qquad (2)$$

tells us that when this happens and the phase goes moving off to the right, a voltage will be generated across the junction. If this does not occur, the state must have been $|0\rangle$.

The end result of all this is somewhat surprising: it is possible to build a macroscopic qubit using normal electronics and fairly standard fabrication techniques. The system can be controlled with ordinary microwave electronics, and coupling several qubits together can be done by tuning them to a resonator, ~~when~~ This provides strong coupling but by changing the bias current the coupling can be turned off.

This has tremendous advantages over, for example, electron spins which require somewhat esoteric measurement systems and do not have tunable coupling.

The primary reference for this was

"Quantum Fluctuations in Electrical Circuits" by Michel H. Devoret

in addition to the paper mentioned on page (1)

"Superconducting Qubits and the Physics of Josephson Junctions" by Kevin Osbourne, John Martinis from the Les Houches conference proceedings