Squeezed Light

Physics 215C Final Presentation Notes

Andrew Merrill

1 Brief Overview and Introduction

Before tackling the main part of this lecture on the topic of squeezed states of light, let's review some of the basics of the quantum mechanics of the electromagnetic field. As you know, in order to describe the many unique and novel features of light, one must use a quantum mechanically valid description of the electromagnetic field.

1.1 Quantized Electromagnetic Field

The nonrelativistic, quantum mechanical description of the electromagnetic field is formulated in terms of the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ which can be written in terms of the canonically conjugate position and momentum operators $\hat{Q}$ and $\hat{P}$ [1].

$$\hat{a}_{\mathbf{k},\alpha} = \sqrt{\frac{\omega}{2\hbar}}\left(\hat{Q}_{\mathbf{k},\alpha} + \frac{i}{\omega}\hat{P}_{\mathbf{k},\alpha}\right) \tag{1.1}$$

$$\hat{a}^\dagger_{\mathbf{k},\alpha} = \sqrt{\frac{\omega}{2\hbar}}\left(\hat{Q}_{\mathbf{k},\alpha} - \frac{i}{\omega}\hat{P}_{\mathbf{k},\alpha}\right) \tag{1.2}$$

$\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relations [1]:

$$\left[\hat{a}_{\mathbf{k},\alpha}, \hat{a}^\dagger_{\mathbf{k}',\alpha'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\alpha\alpha'} \tag{1.3}$$

$$\left[\hat{a}_{\mathbf{k},\alpha}, \hat{a}_{\mathbf{k}',\alpha'}\right] = \left[\hat{a}^\dagger_{\mathbf{k},\alpha}, \hat{a}^\dagger_{\mathbf{k}',\alpha'}\right] = 0 \tag{1.4}$$

Using Eqns. (1.1) and (1.2), the vector potential *operator* $\hat{A}(\mathbf{r},t)$ is written as,

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha}\sqrt{\frac{2\pi\hbar c^2}{V\omega}}\boldsymbol{\epsilon}_{\mathbf{k},\alpha}\left[\hat{a}_{\mathbf{k},\alpha}e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}^\dagger_{\mathbf{k},\alpha}e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \tag{1.5}$$

The electric and magnetic field *operators* are then derived via Maxwell's equations,

$$\hat{\mathbf{E}}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial\hat{\mathbf{A}}}{\partial t} = \sum_{\mathbf{k},\alpha} i\sqrt{\frac{2\pi\hbar\omega}{V}}\boldsymbol{\epsilon}_{\mathbf{k},\alpha}\left[\hat{a}_{\mathbf{k},\alpha}e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^\dagger_{\mathbf{k},\alpha}e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \tag{1.6}$$

$$\hat{\mathbf{B}}(\mathbf{r},t) = \boldsymbol{\nabla}\times\hat{\mathbf{A}} = \sum_{\mathbf{k},\alpha} i\sqrt{\frac{2\pi\hbar c^2}{V\omega}}(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k},\alpha})\left[\hat{a}_{\mathbf{k},\alpha}e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^\dagger_{\mathbf{k},\alpha}e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \tag{1.7}$$

The time dependence of the fields are carried by the operators $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{a}(t) = \hat{a}\,e^{-i\omega t} \tag{1.8a}$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger e^{+i\omega t} \tag{1.8b}$$

Inverting Eqns. (1.1) and (1.2), the canonically conjugate coordinates $\hat{Q}$ and $\hat{P}$ are given by,

$$\hat{Q}_{\mathbf{k},\alpha} = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{a}_{\mathbf{k},\alpha} + \hat{a}^{\dagger}_{\mathbf{k},\alpha}\right) \tag{1.9}$$

$$\hat{P}_{\mathbf{k},\alpha} = -i\sqrt{\frac{\hbar\omega}{2}}\left(\hat{a}_{\mathbf{k},\alpha} - \hat{a}^{\dagger}_{\mathbf{k},\alpha}\right) \tag{1.10}$$

It follows from Eqns. (1.9) and (1.10) that the Hamiltonian for the field is:

$$\begin{aligned}\hat{H} &= \frac{1}{2}\sum_{\mathbf{k},\alpha}\left(\hat{P}^2_{\mathbf{k},\alpha} + \omega^2\hat{Q}^2_{\mathbf{k},\alpha}\right)\\ &= \frac{1}{2}\sum_{\mathbf{k},\alpha}\hbar\omega\left(\hat{a}_{\mathbf{k},\alpha}\hat{a}^{\dagger}_{\mathbf{k},\alpha} + \hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha}\right)\\ &= \sum_{\mathbf{k},\alpha}\hbar\omega\left(\hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha} + \frac{1}{2}\right)\end{aligned} \tag{1.11}$$

and the total momentum of the field becomes:

$$\begin{aligned}\hat{\mathbf{P}} &= \sum_{\mathbf{k},\alpha}\frac{1}{2\omega}\left(\hat{P}^2_{\mathbf{k},\alpha} + \omega^2\hat{Q}^2_{\mathbf{k},\alpha}\right)\mathbf{k}\\ &= \sum_{\mathbf{k},\alpha}\hbar\mathbf{k}\left(\hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha} + \frac{1}{2}\right)\\ &= \sum_{\mathbf{k},\alpha}\hbar\mathbf{k}\hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha}\end{aligned} \tag{1.12}$$

Thus, we see that in transforming from the classical to quantum mechanical description of the field, the classical fields have been replaced by quantum-mechanical operators. Furthermore, just as in the case of the one-dimensional harmonic osciallator, the operators $\hat{a}$ and $\hat{a}^{\dagger}$ act as raising and lowering operators, respectively, and each $(\mathbf{k},\alpha)$ mode of the field acts as an indepdent oscillator. As was done in the case of the harmonic oscillator, the operator $\hat{N}_{\mathbf{k},\alpha} \equiv \hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha}$ is a number operator. It has eigenvalues $n_{\mathbf{k},\alpha}$ which represents the number of photons with wavevector $\mathbf{k}$ and polarization $\epsilon_{\mathbf{k},\alpha}$ in the field. A photon in mode $(\mathbf{k},\alpha)$ has energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$.

2 Coherent and Squeezed States of the Electromagnetic Field

It was demonstrated above that the pure radiation field can be transformed into a set of harmonic oscillators, one for each mode of the field specified by wavevector $\mathbf{k}$ and polarization $\epsilon_{\mathbf{k},\alpha}$. The eigenstates of the number operator $\hat{N}_{\mathbf{k},\alpha}$ are denoted $|n_{\mathbf{k},\alpha}\rangle$, which may be called *number states* or *photon states*. If one attempts to ascertain the wave properties of the field from the number states by computing expectation values of the oscillators "position" and "momentum" operators $\hat{Q}$ and $\hat{P}$[1], one finds:

$$\langle n|\hat{Q}(t)|n\rangle = \langle n|\hat{P}(t)|n\rangle = 0 \tag{2.1}$$

and similarly, for the electric field,

$$\langle n|\hat{E}(\mathbf{r},t)|n\rangle = 0 \tag{2.2}$$

In other words, when the exact number of photons n are defined, the classical oscillatory time-dependence of the field is lost. The n-representation brings about the particle-like aspects of the field embodied by the photon. This description can be thought of as the extreme quantum limit. However, it is sometimes more convenient to adopt a wave-like description of the field with well defined phase and amplitude. We wish to achieve this formulation while still retaining the quantum aspects of the field, and so the quesiton arises, "Do such states exist?". Indeed, such states do exist and are called *coherent states* [2–5].

2.1 Coherent State

Here we consider the normalized eigenstates $|\alpha\rangle$ of the annihilation operator $\hat{a}$, i.e. we investigate the eigenvalue problem:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{2.3}$$

The Hermitian conjugate of Eq. (2.3) is given by

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^* \tag{2.4}$$

The expectation values of $\hat{Q}(t)$ and $\hat{P}(t)$ in the state $|\alpha\rangle$ are

$$\begin{aligned}\langle Q\rangle &= \langle\alpha|\hat{Q}(t)|\alpha\rangle \\ &= \sqrt{\frac{\hbar}{2\omega}}\langle\alpha|\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}|\alpha\rangle \\ &= Q_0\left[\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}\right]\end{aligned} \tag{2.5}$$

and

[1] at this point, to simplify notation, I drop the subscripts $\mathbf{k}$,α from all operators and number states

$$\begin{aligned}\langle\hat{P}\rangle &= \langle\alpha|\hat{P}(t)|\alpha\rangle \\ &= -i\sqrt{\frac{\hbar\omega}{2}}\langle\alpha|\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}|\alpha\rangle \\ &= -i\sqrt{\frac{\hbar\omega}{2}}\left[\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}\right] \\ &= -i\omega Q_0\left[\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}\right] \\ &= \frac{d\langle\hat{Q}\rangle}{dt}\end{aligned} \tag{2.6}$$

where $Q_0 = \sqrt{\frac{\hbar}{2\omega}}$. Writing the complex quantity α in terms of an amplitude and phase, i.e. $\alpha = |\alpha|e^{i\phi}$, the expectation values of $\langle\hat{Q}\rangle$ and $\langle\hat{P}\rangle$ can be written as:

$$\begin{aligned}\langle\hat{Q}\rangle &= Q_0\left[\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}\right] \\ &= Q_0|\alpha|\left[e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right] \\ &= 2Q_0|\alpha|\cos(\omega t - \phi)\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\langle\hat{P}\rangle &= \frac{d\langle\hat{Q}\rangle}{dt} \\ &= -2\omega Q_0|\alpha|\sin(\omega t - \phi)\end{aligned} \tag{2.8}$$

Similarly, in a coherent state, the expectation value of $\hat{\mathbf{E}}$ becomes

$$\begin{aligned}\langle\hat{\mathbf{E}}\rangle &= \langle\alpha|\hat{\mathbf{E}}|\alpha\rangle \\ &= i\sqrt{\frac{2\pi\hbar\omega}{V}}\langle\alpha|\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{a}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}|\alpha\rangle \\ &= i\sqrt{\frac{2\pi\hbar\omega}{V}}\left[\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \alpha^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \\ &= i\sqrt{\frac{2\pi\hbar\omega}{V}}|\alpha|\left[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi)} - e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi)}\right] \\ &= -2|\alpha|\sqrt{\frac{2\pi\hbar\omega}{V}}\sin(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)\end{aligned} \tag{2.9}$$

In a coherent state, the expectation value of n is not specified exactly since

$$\begin{aligned}\langle n\rangle &= \langle\alpha|\hat{N}|\alpha\rangle \\ &= \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle \\ &= |\alpha|^2\end{aligned} \tag{2.10}$$

Also,

$$\begin{aligned}
\langle n^2\rangle &= \langle\alpha|\hat{N}^2|\alpha\rangle \\
&= \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle \\
&= |\alpha|^2\langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle \\
&= |\alpha|^2\langle\alpha|(1+\hat{a}^\dagger\hat{a})|\alpha\rangle \\
&= |\alpha|^2+|\alpha|^4
\end{aligned} \tag{2.11}$$

Thus, the variance in n is equal to

$$\begin{aligned}
(\Delta n)^2 &= \langle n^2\rangle - \langle n\rangle^2 \\
&= |\alpha|^2 \\
&= \langle n\rangle
\end{aligned} \tag{2.12}$$

Since the number states for a complete set, the coherent states can be expanded as,

$$|\alpha\rangle = \sum_{n=0}^{\infty}|n\rangle\langle n|\alpha\rangle = \sum_{n=0}^{\infty}c_n(\alpha)|n\rangle \tag{2.13}$$

where $|c_n(\alpha)|^2 = |\langle n|\alpha\rangle|^2$ is the probability that a measurement will find the oscillator in state $|n\rangle$. By applying $\hat{a}$ to $|\alpha\rangle$ we find,

$$\begin{aligned}
\hat{a}|\alpha\rangle &= \sum_{n=0} c_n(\alpha)\hat{a}|n\rangle \\
&= c_0(\alpha)\underbrace{\hat{a}|0\rangle}_{0} + \sum_{n=1} c_n(\alpha)\hat{a}|n\rangle \\
&= \sum_{n=1} c_n(\alpha)\sqrt{n}|n-1\rangle \\
&= \sum_{n=0} c_{n+1}(\alpha)\sqrt{n+1}|n\rangle
\end{aligned} \tag{2.14}$$

But, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$so this implies,

$$\begin{aligned}
\alpha\sum_{n=0} c_n(\alpha)|n\rangle &= \sum_{n=0} c_{n+1}(\alpha)\sqrt{n+1}|n\rangle \\
\Rightarrow \alpha c_n(\alpha) &= c_{n+1}(\alpha)\sqrt{n+1} \\
\Rightarrow c_{n+1} &= \frac{\alpha}{\sqrt{n+1}}c_n(\alpha)
\end{aligned} \tag{2.15}$$

Writing the first few terms of $c_n(\alpha)$ for $n = 0, 1, 2, \ldots$, we find,

$$\begin{aligned}
c_1(\alpha) &= \frac{\alpha}{\sqrt{1}}c_0(\alpha) \\
c_2(\alpha) &= \frac{\alpha}{\sqrt{2}}c_1(\alpha) = \frac{\alpha}{\sqrt{1}}\frac{\alpha}{\sqrt{2}}c_0(\alpha) \\
c_3(\alpha) &= \frac{\alpha}{\sqrt{3}}c_2(\alpha) = \frac{\alpha}{\sqrt{1}}\frac{\alpha}{\sqrt{2}}\frac{\alpha}{\sqrt{3}}c_0(\alpha) \\
&\vdots \\
c_n(\alpha) &= \frac{\alpha^n}{\sqrt{n!}}c_0(\alpha)
\end{aligned} \tag{2.16}$$

$c_0(\alpha)$ is determined by normalization of $\langle\alpha|\alpha\rangle = 1$ to be $c_0 = e^{-\frac{1}{2}|\alpha|^2}$. Thus Eq. (2.13) becomes,

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.17}$$

Using these results, the probability $P(n) = |c_n(\alpha)|^2 = |\langle n|\alpha\rangle|^2$ becomes,

$$\begin{aligned}
P(n) &= |\langle n|\alpha\rangle|^2 \\
&= \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} \\
&= \frac{\langle n\rangle^n}{n!}e^{-\langle n\rangle}
\end{aligned} \tag{2.18}$$

which is a Poisson distribution. The Poisson distribution is unique in that it has no free parameters.
The variance of $\hat{Q}$ and $\hat{P}$ in a coherent state are given by

$$\begin{aligned}
(\Delta Q)^2 &= \langle Q^2\rangle - \langle Q\rangle^2 \\
&= \langle\alpha|\hat{Q}^2|\alpha\rangle - \langle\alpha|\hat{Q}|\alpha\rangle^2 \\
&= \langle\alpha|\frac{\hbar}{2\omega}\left(\hat{a}+\hat{a}^\dagger\right)^2|\alpha\rangle - \langle\alpha|\sqrt{\frac{\hbar}{2\omega}}\left(\hat{a}+\hat{a}^\dagger\right)|\alpha\rangle^2 \\
&= \frac{\hbar}{2\omega}\langle\alpha|\left(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}\right)|\alpha\rangle - \frac{\hbar}{2\omega}\langle\alpha|\left(\hat{a}+\hat{a}^\dagger\right)|\alpha\rangle^2 \\
&= \frac{\hbar}{2\omega}\left(\alpha^2+1+2\alpha\alpha^*+\alpha^{*2}\right) - \frac{\hbar}{2\omega}\left(\alpha^2+2\alpha\alpha^*+\alpha^{*2}\right) \\
&= \frac{\hbar}{2\omega}
\end{aligned} \tag{2.19}$$

and

$$
\begin{aligned}
(\Delta P)^2 &= \langle P^2\rangle - \langle P\rangle^2 \\
&= \langle\alpha|\hat{P}^2|\alpha\rangle - \langle\alpha|\hat{P}|\alpha\rangle^2 \\
&= \langle\alpha|-\frac{\hbar\omega}{2}\left(\hat{a}-\hat{a}^\dagger\right)^2|\alpha\rangle - \langle\alpha|-i\sqrt{\frac{\hbar\omega}{2}}\left(\hat{a}-\hat{a}^\dagger\right)|\alpha\rangle^2 \\
&= -\frac{\hbar\omega}{2}\langle\alpha|\left(\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}\right)|\alpha\rangle + \frac{\hbar\omega}{2}\langle\alpha|\left(\hat{a}-\hat{a}^\dagger\right)|\alpha\rangle^2 \\
&= -\frac{\hbar\omega}{2}\left(\alpha^2-1-2\alpha\alpha^*+\alpha^{*2}\right) + \frac{\hbar\omega}{2}\left(\alpha^2-2\alpha\alpha^*+\alpha^{*2}\right) \\
&= \frac{\hbar\omega}{2}
\end{aligned}
\tag{2.20}
$$

Thus, we immediately see that coherent states are miniumum uncertainty states satisfying,

$$
\Delta Q\Delta P = \sqrt{\frac{\hbar}{2\omega}}\sqrt{\frac{\hbar\omega}{2}} = \frac{\hbar}{2} \tag{2.21}
$$

2.1.1 Wave Functions of Coherent States

$$
\hat{a}(t) = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{Q}+i\hat{P}\right) \tag{2.22}
$$

$\hat{a}(t)|\alpha\rangle = \hat{a}e^{-i\omega t}|\alpha\rangle = \alpha e^{-i\omega t}|\alpha\rangle$

$$
\frac{1}{2\hbar\omega}\langle Q|\left(\omega\hat{Q}+i\hat{P}\right)|\alpha\rangle = \alpha e^{-i\omega t}\langle Q|\alpha\rangle \tag{2.23}
$$

where $\langle Q|\alpha\rangle$ is the coherent state wavefunction $\psi_\alpha(Q,t)$. Using $\langle Q|\hat{Q}|\alpha\rangle = Q\psi_\alpha(Q,t)$ and $\langle Q|\hat{P}|\alpha\rangle = -i\hbar\frac{\partial}{\partial Q}\psi_\alpha(Q,t)$ produces the following differential equation for the wavefunction,

$$
\frac{\partial}{\partial Q}\psi_\alpha(Q,t) = \left(\sqrt{\frac{2\omega}{\hbar}}\alpha e^{-i\omega t} - \frac{\omega}{\hbar}Q\right)\psi_\alpha(Q,t) \tag{2.24}
$$

Integrating equation (2.24) yields,

$$
\psi_\alpha(Q,t) = C(\alpha,t)\exp\left(-\frac{\omega}{2\hbar}Q^2 + \sqrt{\frac{2\omega}{\hbar}}\alpha e^{-i\omega t}Q\right) \tag{2.25}
$$

where $C(\alpha,t)$ is determined by normalization to be

$$
C(\alpha,t) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-|\alpha|^2\cos^2(\omega t-\phi)} \tag{2.26}
$$

The position-space probability density is given by,

$$
|\psi_\alpha(Q,t)|^2 = \sqrt{\frac{\omega}{\pi\hbar}}e^{-\frac{\omega}{\hbar}\left(Q-\sqrt{\frac{2\hbar}{\omega}}|\alpha|\cos(\omega t-\phi)\right)^2} \tag{2.27}
$$

2.1.2 The Displacement Operator

Coherent states can be generated, mathematically, via a "displacement" operator

$$D(\alpha) \equiv e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \tag{2.28}$$

$$D^{-1}(\alpha)\hat{a}D(\alpha) = e^{\alpha^*\hat{a}}e^{-\alpha\hat{a}^\dagger}\hat{a}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} \tag{2.29}$$

$$e^{-\alpha\hat{A}}\hat{B}e^{\alpha\hat{A}} = \hat{B} - \alpha\left[\hat{A},\hat{B}\right] + \frac{\alpha^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \cdots \tag{2.30}$$

$$e^{-\alpha\hat{a}^\dagger}\hat{a}e^{\alpha\hat{a}^\dagger} = \hat{a} + \alpha \tag{2.31}$$

3 Squeezed States of Light

$$\alpha = \alpha_1 + i\alpha_2 \tag{3.1}$$

where $\alpha_1 = |\alpha| \cos\phi$ and $\alpha_2 = |\alpha| \sin\phi$.

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) = \sqrt{\frac{\omega}{2\hbar}}\hat{Q} \tag{3.2}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) = \frac{1}{\sqrt{2\hbar\omega}}\hat{P} \tag{3.3}$$

The eigenstates of Eqs. (3.2) and (3.3) are $|x_1\rangle$ and $|x_2\rangle$, respectively. The expectation values are given by:

$$\langle x_1\rangle = \langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2}\left(\langle\alpha|\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^\dagger|\alpha\rangle\right) = \alpha_1 \tag{3.4}$$

$$\langle x_2\rangle = \langle\alpha|\hat{X}_2|\alpha\rangle = \frac{1}{2i}\left(\langle\alpha|\hat{a}|\alpha\rangle - \langle\alpha|\hat{a}^\dagger|\alpha\rangle\right) = \alpha_2 \tag{3.5}$$

$$\Delta x_1 = \sqrt{\frac{\omega}{2\hbar}}\Delta Q = \sqrt{\frac{\omega}{2\hbar}}\sqrt{\frac{\hbar}{2\omega}} = \frac{1}{2} \tag{3.6}$$

$$\Delta x_2 = \frac{1}{\sqrt{2\hbar\omega}}\Delta P = \frac{1}{\sqrt{2\hbar\omega}}\sqrt{\frac{\hbar\omega}{2}} = \frac{1}{2} \tag{3.7}$$

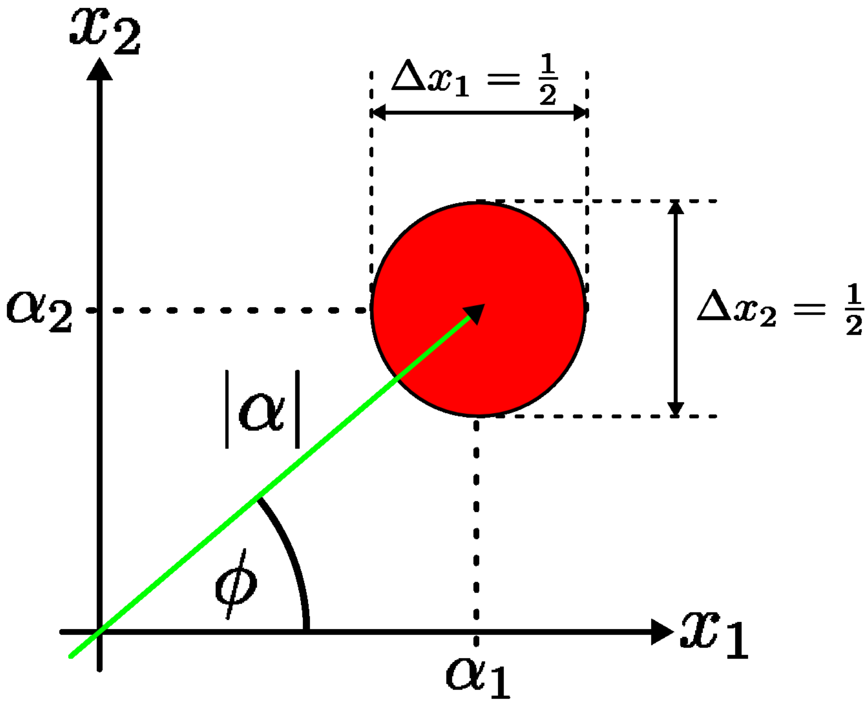

Figure 1: Phase space representation of coherent state

$$\left[\hat{X}_1, \hat{X}_2\right] = \frac{1}{2\hbar}\left[\hat{Q}, \hat{P}\right] = \frac{i}{2} \tag{3.8}$$

$$\Delta x_1 \Delta x_2 \geq \frac{1}{4} \tag{3.9}$$

3.1 Examples of Squeezed States

3.1.1 Scaling squeezed states

Scaling squeezed states are denoted by the state ket $|\alpha_{\rm ss}; \mu, \nu\rangle$. These states are generated by applying a *squeezing operator* $\hat{S}(\mu, \nu)$ to a coherent state $|\alpha_{\rm in}\rangle$, i.e.

$$|\alpha_{\rm ss}; \mu, \nu\rangle = \hat{S}(\mu, \nu)|\alpha_{\rm in}\rangle \tag{3.10}$$

$\hat{S}(\mu, \nu)$ is defined as:

$$\hat{S}(\mu, \nu) = e^{\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})} \tag{3.11}$$

where

$$\zeta = s\, e^{i\theta}. \tag{3.12}$$

s and θ are called the *degree of squeezing* and *squeezing angle*, respectively. The *squeezing parameters* μ and ν are given by:

$$\mu = \cosh s \qquad \text{and} \qquad \nu = e^{i\theta} \sinh s. \tag{3.13}$$

$\alpha_{\rm in} = |\alpha_{\rm in}| e^{i\phi_{\rm in}}$

$$\alpha_{\rm ss} = |\alpha_{\rm ss}| e^{i\phi_{\rm ss}} = \mu \alpha_{\rm in} + \nu e^{i\theta} \alpha_{\rm in}^* \tag{3.14}$$

3.1.2 Coherent squeezed states

Coherent squeezed states are denoted by the state ket $|\alpha_{\rm cs}; \mu, \nu\rangle$ are generated by the following transformation:

$$|\alpha_{\rm cs}; \mu, \nu\rangle = \hat{D}(\alpha_{\rm cs})\hat{S}(\mu, \nu)|0\rangle = \hat{D}(\alpha_{\rm cs})|0; \mu, \nu\rangle \tag{3.15}$$

where $\hat{D}(\alpha_{\rm cs})$ is the unitary transformation operator defined by Eq. (2.28). In words, a coherent squeezed state is obtained by first generating a scaled squeezed state of the vacuum ket $|0\rangle$, which in phase space appears as an uncertainty ellipse centered at the origin, resulting in a squeezed vacuum state $|0; \mu, \nu\rangle$. Next, the operator $D(\alpha_{\rm cs})$ displaces the center of the ellipse to the amplitude $\alpha_{\rm cs} = |\alpha_{\rm cs}| e^{i\phi_{\rm cs}}$.

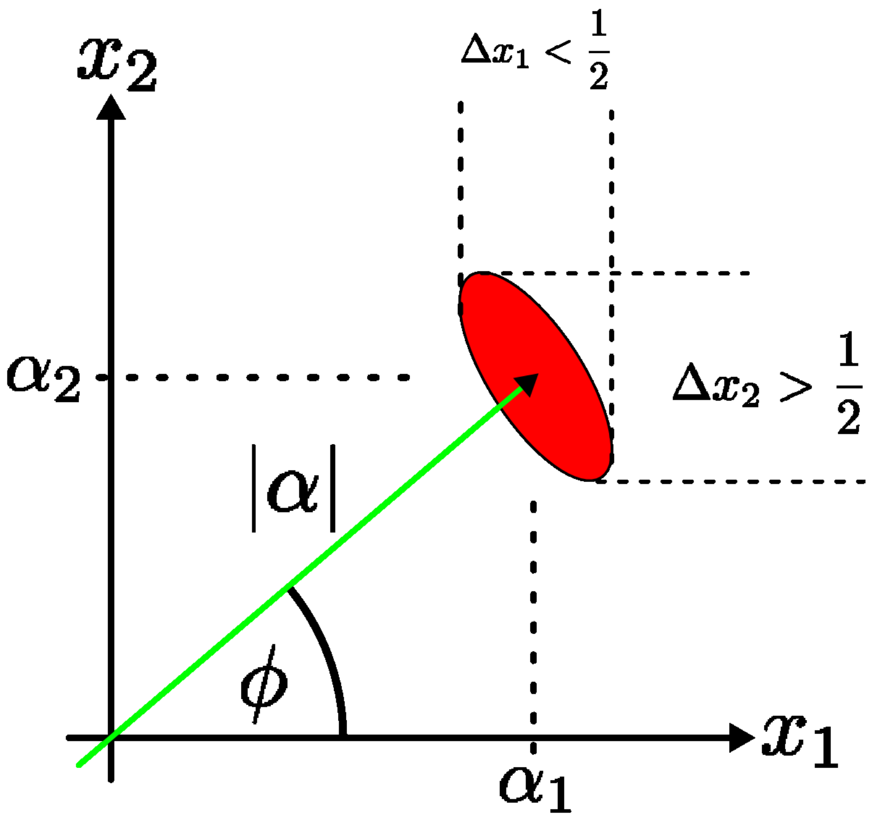

Figure 2: Phase space representation of squeezed state uncertainty ellipse.

3.2 Properties of Squeezed States

When the squeezing angle θ takes on the value of 0 or π, the two axes of the uncertainty ellipse like parallel to the coordinate axes. The noise components, Δx_1 and Δx_2 are given by the lengths of the semimajor and semiminor axes, $\frac{1}{2}e^{-s}$ and $\frac{1}{2}e^{s}$, respectively. Thus, one has $\Delta x_1 \Delta x_2 = \frac{1}{4}$ which implies that squeezed states of this type are minimum uncertainty states. For θ not equal to 0 or π, the squeezed states do not satisfy this condition.

3.2.1 Photon counting statistics

4 Generation and Applications of Squeezed Light

This is where I ran out of time.

4.1 Generation

- Squeezed light can be generated by passing light through as series of optical crystals, cavity resonators, and other nonlinear optical techniques.

4.2 Applications

- Applications include gravitational wave detection, secure data transmission, quantum computation, quantum memory, etc.

4.2.1 Detection of Gravity Waves

As you know, an accelerating charge emits electromagnetic radiation. Similarly, Einstein's theory of general relativity predicts that accelerating mass emits gravitational radiation, or gravity waves. Gravity waves can be detected with gravitational interferometers.

- Sensitivity of gravitational interferometers limited by "shot noise" or *quantum noise*.
- Fluctuations in photon number could trigger false positives
- Quantum noise can be reduced by squeezing light.
- Advanced LIGO, GEO 600, VIRGO, and other gravitational wave interferometers are in the process of implementing squeezed light detection schemes.
- If Einstein was right, we should hear about the direct detection of gravity waves in the near future.

A schematic of a gravity wave interferometer is shown in Fig. 3. The desired sensitivity is $\frac{\Delta L}{L} \approx 10^{-21}$. To get a feel for the length scales necessary, consider the following values in Table 1 [6].

Table 1: A look at ΔL vs. L

ΔL		L
$\Delta L \approx \frac{1}{10}$mm	$\Rightarrow$	$L \approx 3$ light years
$\Delta L \approx 10^{-13}$cm	$\Rightarrow$	$L \approx 4000$ km
$\Delta L \approx 10^{-16}$cm	$\Rightarrow$	$L \approx 4$ km

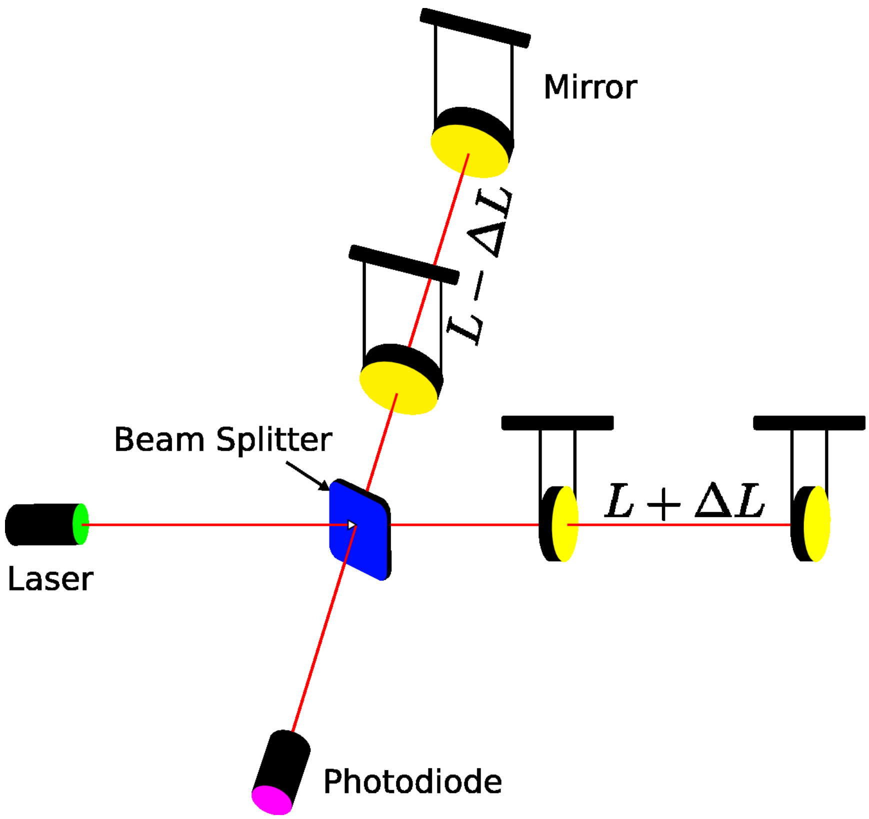

Figure 3: Schematic of gravitational wave interferometer.

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