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Phys 215C Final

The deuteron, N-P scattering, and the strong force

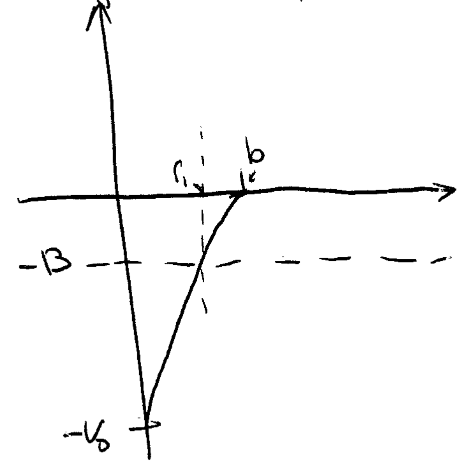
there are two questions I would like to ask:
(Q1) - what can QM tell us about the deuteron?
(Q2) - what can the deuteron then tell us about the strong force?

1. Let's start with the basics: the deuteron is the only existing bound state of two "nucleons". It consists of a neutron and a proton in the ground state of the ~~xxx~~ system. The deuteron has no excited states. The deuteron is also a triplet spin state; no singlet has been experimentally found, and later on we will try to explain this.

For now, we may want to ask - what makes the deuteron interesting? Two answers come to mind:
- the deuteron seems to be the simplest naturally occurring probe of the strong force; - more specifically, the strong force as applied to nuclear structure.
- the deuteron would likely not exist without quantum mechanics; classically the bound state is essentially impossible.

Let's justify the latter claim with some actual physics.

Assume $M_p \sim M_n = M$ then the problem is essentially just a 2-body system held together by an attractive, short range force. For now we will assume that we are dealing with a central force, which is naturally provided by a potential $V(r)$, which is negative within some short range $r<b$, and negligible outside.



Write down Schrodinger's Eqn:

$$-\left(\frac{\hbar^2}{2\mu}\right)\nabla^2\Psi(\vec{r}) + V(r)\Psi(\vec{r}) = E\Psi(\vec{r})$$

here μ = reduced mass $= \frac{1}{2}M$.

The ground state should be spherically symmetric $\Rightarrow$
$\Rightarrow$ write $\Psi(\vec{r}) = \Psi(r) = u(r)/r$

Then:

$$-\left(\frac{\hbar^2}{M}\right)\frac{d^2u}{dr^2} + V(r)u(r) = Eu(r) \quad \text{, or}$$

$$\frac{d^2u}{dr^2} + k^2(r)u(r) = 0, \quad \text{where } k(r) = \pm\frac{1}{\hbar}\left[M(E-V(r))\right]^{1/2}$$

We are interested in a solution for which $E = -B$ (the binding energy). As evident in the above picture, there will be a point $r=r_1$ at which $V(r_1) = E$.

$k(r)$ is real inside r_1, imaginary outside. $r<r_1$ is the classically "allowed" zone, while $r>r_1$ is forbidden.

In the forbidden zone, V rapidly becomes negligible $\Rightarrow$ neglect it in favor of E.
then we have $k_{out} = \pm \frac{1}{\hbar}[-MB]^{1/2} = \pm \frac{i}{\hbar}(MB)^{1/2}$.

then $\Psi_{out} \propto \exp(-r/R)$ (exponential decay)

where $R = |k|^{-1} = \hbar(MB)^{-1/2}$.

now, $B = 2.2$ MeV. This gives us a value of

$$\boxed{R = 4.31 \times 10^{-13}\text{ cm}}$$ this is the effective radius of the deuteron!

And here is the remarkable thing - this radius is more than twice the effective range of the force ($b \sim 2 \times 10^{-13}$ cm). Hence, the deuteron spends at least as much time in the forbidden zone as in the allowed zone - and that is why I say the deuteron is not 'classically' viable!

2. We would now like to ask what the deuteron alone can tell us about the nuclear potential - in particular, we want to know if it sheds any light on the depth of the potential well
To simplify matters, assume the potential well is a spherical square well with $V = \begin{cases} -V_0 & r < b \\ 0 & r > b \end{cases}$

then we can immediately write down the interior and exterior solutions for the ground state E

$$u(r) = \begin{cases} A \sin kr, & k = [M(V_0 - B)]^{1/2}/\hbar \quad r<b \\ B e^{-\kappa r}, & \kappa = \sqrt{MB}/\hbar \quad r>b \end{cases}$$

Matching at the boundary, we get:

$$\tan kb = -\left(\frac{V_0 - B}{B}\right)^{1/2}$$

Now, suppose ~~that~~ $V_0 \gg B$ [we will need to check this later].

then the RHS is large, hence

$kb \sim \frac{\pi}{2}$ (note this implies that only about 1/4 of a wavelength fits into the interior of the potential. this is the minimal condition for having a bound state).

So,

$$\sqrt{\frac{MV_0}{\hbar^2}} \sim \frac{\pi}{2b}$$

$$\Rightarrow \boxed{V_0 b^2 \sim \left(\frac{\pi}{2}\right)^2 \frac{\hbar^2}{M} \sim 1 \text{ MeV barn}}$$

Using $b \sim 2\text{fm}$ as before, we get

$$\boxed{V_0 \sim 36 \text{ MeV}}$$ (note this justifies our $V_0 \gg B$ assumption)

So, just from the knowledge of the binding energy of the deuteron we have arrived at the approximate depth of the nuclear potential.

That's not really true, of course - where did our knowledge of this "b" parameter come from? Without it, this entire approach falls apart. The answer is that, it comes from neutron-proton scattering experiments, to which we now turn our attention. We will not derive b directly, but we will attempt to solve for the differential cross section and see if our solution agrees with experiment.

3. Before we can do this, however, a quick review of s-wave "zero" energy scattering is in order. (for a full discussion, see Prof Blaes' notes pp 751 -)

You'll no doubt recall that the "s-wave" term in the partial wave expansion for $f(\theta)$ is:

$$f_0 = \frac{1}{k} e^{i\delta_0} \sin\delta_0$$

In "zero" energy scattering, we claim this is <u>the</u> dominant term in the expansion and discard all the others. In fact, we go even further: we take the limit as $k \to 0$.

To this end, a quantity called the "scattering length", denoted by "a", becomes very useful:

$$a = -\lim_{k \to 0} \frac{\tan\delta_0}{k}$$

then

$$\lim_{k\to 0} f(\theta) = \frac{1}{k} e^{i\delta}\sin\delta \to \frac{-a}{1+ika} \to -a$$

and then $\boxed{\sigma = 4\pi a^2}$

this explains the name scattering length: it is the effective radius of a hard sphere that would produce the same total cross-section.

In addition, a connects nicely with other quantities in the $k\to 0$ limit:

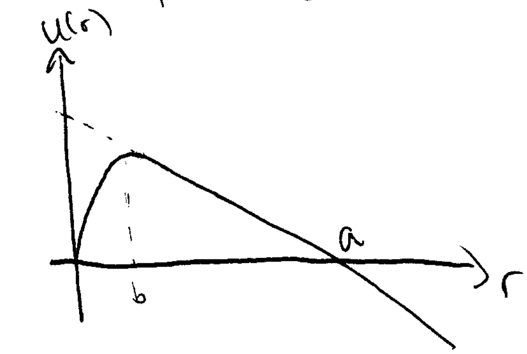
in this limit the wavefunction outside the well becomes $\psi \sim \frac{r-a}{r} \Rightarrow u(r) \sim (r-a)$, a line!

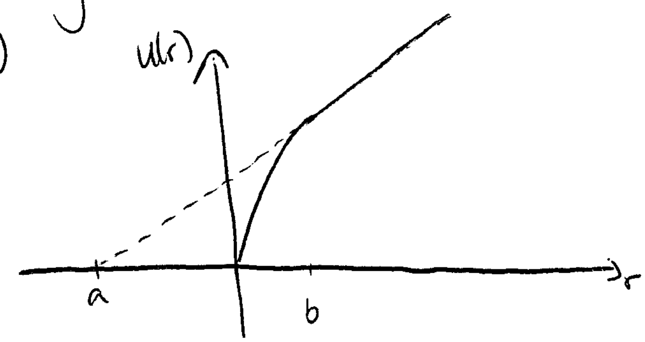
thus all solutions $u(r)$ in this limit tend to a line as r gets large, and this line intersects the r-axis at $r=a$.

Finally, the sign of a turns out to be intrinsically ~~___~~ related to bound states. To understand this, note that since $\psi = u(r)/r$, then

$|\psi|^2 = \frac{u(r)^2}{r^2}$, which means $|u(r)|^2 dr$ becomes the probability of finding the two particles (in our case) within $r+dr$ of each other. then, plotting $u(r)$ is essentially equivalent to plotting probability amplitude for being at a certain r.

Now, consider the following 2 cases:

a) 

b) 

In (a), we obviously see the bulk of probability amplitude lying inside or close to $r=b$. This is a bound state.

On the other hand (b) exhibits an increase in $u(r)$ with increasing r and thus is not a bound wavefunction. thus,

$a > 0$ corresponds to bound states

while $a < 0$ — " — " free states.

We now have enough information to calculate the total cross-section for neutron-proton scattering.

4. "s-wave" "zero" energy N-P scattering

Assume, as before, a square well potential of depth V_0 and range b. This time, however, we are considering solutions with $E > 0$.

$$\text{then} \quad u(r) = \begin{cases} B\sin(k'r) & r < b \\ C\sin(kr + \delta_0) & r > b \end{cases}$$

where $k' = \frac{1}{\hbar}[M(V_0 + E)]^{1/2}$ and $k = [ME]^{1/2}/\hbar$.

Matching at interface, we have:

$$k'\cot(k'b) = k\cot(kb + \delta_0)$$

now recall that $a = -\lim_{k \to 0} \frac{\tan\delta_0}{k}$

then $\delta_0 \sim -ak$

(note b/c we want δ_0 to stay finite as $k \to 0$, we must also have $\delta_0 \to 0$).

Assume $a \gg b$, then we have the RHS $\Rightarrow$

$\Rightarrow k\cot(\delta_0)$.

On the LHS, ~~assume~~ assume $V_0 \gg E$, then

$$k' = \frac{1}{\hbar}[MV]^{1/2}.$$

But this is the K we had when considering the deuteron! (see page 4) then, before we had:

$$k \cot kb = -\sqrt{MB}/\hbar$$

Now we have:

$$k \cot \delta_0 = -\sqrt{MB}/\hbar$$

note this implies that $a = R$!!

then $\quad \sigma(\theta) = \frac{1}{k^2}\sin^2\delta = \frac{1}{k^2}\frac{1}{1+\cot^2\delta_0}$

so $\quad \sigma(\theta) \quad \frac{1}{k^2 + MB/\hbar^2} = \frac{\hbar^2}{M}\frac{1}{E+B} \approx \frac{\hbar^2}{MB}$

(the last approx we put on b/c we mostly care about the range where $E \ll B$).

Plugging in for the constants, we get

theory says : $\boxed{\sigma_{TOT} = 2.3 \text{ barns}}$

However, experiment says:

$$\sigma_{TOT} = 20.4 \text{ barns}!!$$

What is going on?

5. the explanation for this discrepancy between theory and experiment has to do with something I have mentioned before: there is ~~not~~ <u>no</u> bound singlet state of 2 nucleons.

therefore, in 1933 E.P. Wigner suggested that it must be that the nuclear potential is <u>different</u> depending on the spins of the interacting particles. In particular, particles with their spins aligned would see a <u>deeper</u> potential well than those with their spins anti-aligned.

Now of course in a scattering experiment, nobody controlled for the spin variable. therefore we must instead write:

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s$$

the σ_t piece we know; it ~~...~~ implies

$$54 < \sigma_s < 75 \text{ barns}.$$

thus, we have discovered that the nuclear potential is quite obviously spin-<u>dependent</u>!

6. Finally, we'd like to prove ~~once~~ once and for all that no singlet bound state of 2 nucleons exists. this turns out not to be terribly difficult.

Introduce projection operators for the amplitude of the scattered wave.

$$\Pi_s = \frac{1}{4}\left[1 - (\sigma_n \cdot \sigma_p)\right] = \begin{cases} 1 & \text{singlet} \\ 0 & \text{triplet} \end{cases}$$

$$\Pi_t = \frac{1}{4}\left[3 + (\sigma_n \cdot \sigma_p)\right] = \begin{cases} 0 & \text{singlet} \\ 1 & \text{triplet.} \end{cases}$$

then the scattering length operator now becomes:

$$a_{eff} = a_s \Pi_s + a_t \Pi_t .$$

Apply this to parahydrogen molecules (b/c it turns out these can be separated from ortho hydrogen and thus this experiment can be carried out).

in the limit of very slow neutrons, we have

$$a_{sc} = \underbrace{a_{eff,1} + a_{eff,2}}_{\text{the 2 protons of } H_2} = 2\left(\frac{3}{4}a_t + \frac{1}{4}a_s\right) + \cancel{2\left(\frac{1}{4}a_t - \frac{1}{4}a_s\right)(\sigma_n \cdot S)}$$

0

then $\sigma_{para} \sim \lambda 4\pi a_{sc}^2 = \frac{16}{9} 4\pi \left(2\left[\frac{3}{4}a_t + \frac{1}{4}a_s\right]\right)^2$

correction for scattering off a molecule then, as you can see, depending on whether the singlet scattering length a_s is positive or negative, our theory predicts vastly different results. In fact, plugging in the numbers, we have

$\sigma_{para} = 77$ barn if singlet bound state exists

$\sigma_{para} = 7$ barn if it does not.

What do the experiments say?

$\sigma_{para} = 4$ barn!!

this is plenty evidence to suggest that

the singlet bound state is indeed ruled out!, this

implies that the nuclear potential is not just quantitatively, but qualitatively different for different spin configurations.