On the Representation Theory of the Lorentz Group

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The Lorentz Group and $O(3,1)$

The Orthogonal Groups $O\left(3\right)$ and $SO\left(3\right)$

Consider the linear transformation on $\mathbb{R}^3$ defined in the canonical basis as

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

In practice this is the identity map on $\mathbb{R}^3$ and it is used to define the Euclidean metric. Hence we may define the quadratic form

$$\begin{aligned} ds^2 &= \delta_{ij} dx^i dx^j & (2) \\ &= dx_i dx^i & (3) \end{aligned}$$

where I have used the summation convention on repeated indices. The symmetry group of this quadratic form is well known and is denoted $O(3)$. We may define $O(3)$ explicitly as that subgroup of $Gl(3)$ such that$\forall O \in O(3)$ and $\forall v \in \mathbb{R}^3$

$$(Ov)^{T} \cdot \delta(Ov) = v^{T} \delta v. \tag{4}$$

Ergo, $O(3) = \left\{O \in Gl\left(3\right) \mid v \in \mathbb{R}^3 \implies (Ov)^{T} \delta(Ov) = v^{T} \delta v\right\}$ and hence

$$\begin{aligned} \delta &= O^{T} \delta O & (5) \\ &= O^{T} O. & (6) \end{aligned}$$

Therefore, $O \in O\left(3\right)$ satisfies the relations

$$O^{T} = O^{-1} \tag{7}$$

and

$$\det(O) = \pm 1. \tag{8}$$

We furthermore define $SO\left(3\right) = \left\{S \in O\left(3\right) \mid \det S = 1\right\}$.Here we have a fairly important nesting of subgroups. $Gl\left(3\right)$ represents the set of automorphisms of $\mathbb{R}^3$ in that it carries basis sets to basis sets. $O(3)$ which is a subgroups of $Gl\left(3\right)$ carries basis sets to basis sets without changing the norm of each of the vectors in the basis set. Finally, $SO\left(3\right)$ which is itself a subgroup of $O(3)$ preserves not only the norm of the basis vectors in the initial basis set but preserves the chirality of the initial basis set. Furthermore, $SO\left(3\right)$ is a Lie Group with associated Lie Algebra $so\left(3\right)$.

The Generalized Lorentz group

We may generalize the notion of orthogonal group to quadratic forms of various signatures as follows. $O(p,q)$ is the symmetry group of the quadratic form

$$ds^2 = -\left(dx^1\right)^2 - \dots - \left(dx^q\right)^2 + \left(dx^{q+1}\right)^2 + \dots + \left(dx^n\right)^2 \tag{9}$$

where

$$p = n - q. \tag{10}$$

As we all know from Einstein circa 1905 the space-time manifold we live in, $M^4 = \mathbb{R}^4$, consists of the set of all four vectors $v = (ct, x, y, z)$ where we represent $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$.The metric on this space is the Minkowski metric, $\eta \in Gl\,(4)$, given as

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

which induces the quadratic form

$$ds^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2. \tag{12}$$

This form is clearly of signature $(3,1)$ and hence it has as its symmetry group $O\,(3,1)$.In special relativity the transformations which carries us from one basis in M^4 to another are called Lorentz transformations and hence we are able to identify the General Lorentz Group with the group $O\,(3,1)$.

Let $\Lambda \in O\,(3,1)$, then clearly

$$\Lambda^{\mathsf{T}} \eta \Lambda = \eta \tag{13}$$

implies that

$$\Lambda_{00}^2 - \left(\Lambda_{10}^2 + \Lambda_{20}^2 + \Lambda_{30}^2\right) = 1. \tag{14}$$

Ergo,

$$\Lambda_{00}^2 \geq 1 \tag{15}$$

and it follows that

$$\Lambda_{00} \geq 1 \text{ or } \Lambda_{00} \leq -1. \tag{16}$$

Hence we may decompose $O\,(3,1)$ into two subsets

$$\begin{aligned} L^{\uparrow} &= \{\Lambda \in O\,(3,1) \mid \Lambda_{00} \geq 1\} && (17) \\ L^{\downarrow} &= \{\Lambda \in O\,(3,1) \mid \Lambda_{00} \leq 1\}. && (18) \end{aligned}$$

$L^{\uparrow}$ is the set of Orthochronous Lorentz Transformations and they preserve the direction of time and $L^{\uparrow}$ is the set of Anti-Orthochronous Lorentz Transformations and they reverse the direction of time. Moreover, $L^{\uparrow}$ is a subgroup of $O\,(3,1)$ called the Orthochronous Lorentz Group while $L^{\downarrow}$ is not as the identity transformation is Orthochronous. We may furthermore restrict the $L^{\uparrow}$ by taking the determinant of both sides of (13). Hence,

$$\det\left(\Lambda^{\mathsf{T}} \eta \Lambda\right) = \det \eta \tag{19}$$

implies

$$\left(\det \Lambda\right)^2 = 1 \tag{20}$$

or

$$\det \Lambda = \pm 1. \tag{21}$$

Therefore, we may define the Proper Orthochronous Lorentz Group $L_+^{\uparrow}$ as

$$L_+^{\uparrow} = \{\Lambda \in O\,(3,1) \mid \Lambda_{00} \geq 1,\ \det \Lambda = 1\}. \tag{22}$$

Claim 1 *Out of the four possible subsets of* $O\,(3,1)$ *characterized by the determinant and* Λ_{00} *only* $L_+^{\uparrow}$ *is a subgroup.*

Proof. If $\Lambda, \Lambda^* \in L_+^{\uparrow}$, then clearly

$$\left(\Lambda\Lambda^*\right)_{00} = \Lambda_{00}\Lambda_{00}^* + \Lambda_{10}\Lambda_{10}^* + \Lambda_{20}\Lambda_{20}^* + \Lambda_{30}\Lambda_{30}^*. \tag{23}$$

Therefore, by (14) and the Cauchy-Schwartz inequality

$$\begin{aligned} \Lambda_{10}\Lambda_{10}^* + \Lambda_{20}\Lambda_{20}^* + \Lambda_{30}\Lambda_{30}^* &\leq \left(\Lambda_{10}^2 + \Lambda_{20}^2 + \Lambda_{30}^2\right)\left(\Lambda_{10}^{*2} + \Lambda_{20}^{*2} + \Lambda_{30}^{*2}\right) && (24) \\ &< \Lambda_{00}^2 \Lambda_{00}^{*2} && (25) \end{aligned}$$

and the claim follows as Λ_{00} and Λ^*_{00} are greater than unity and $I \in L^\uparrow_+$. ■

The standard group notation for $L^\uparrow_+$ is given by $SO(3,1)^\uparrow$ .Consider the following elements of $Gl(4)$

$$G = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

$H = -G$, $-I$, and I. We may construct all the subsets we've discussed as follows from $SO(3,1)^\uparrow$

$$L^\downarrow_- = GSO(3,1)^\uparrow \tag{27}$$
$$L^\uparrow_- = HSO(3,1)^\uparrow \tag{28}$$
$$L^\downarrow_+ = -ISO(3,1)^\uparrow . \tag{29}$$

Thus, we are now able to see $O(3,1)$ as

$$O(3,1) = ISO(3,1)^\uparrow \cup HSO(3,1)^\uparrow \cup GSO(3,1)^\uparrow \cup -ISO(3,1)^\uparrow . \tag{30}$$

$SO(4)$ and the Lorentz Group

The Representation Theory of $SO(4)$

Consider the quadratic form defined by

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu , \tag{31}$$

where $\delta_{\mu\nu}$ is the Identity transformation on $\mathbb{R}^4$ in the canonical basis. Clearly the symmetry group of (31) is $O(4)$. We wish to construct the irreducible representations of the Proper Orthochronous Lorentz group by considering the irreducible representations of a subgroup, $SO(4)$, of this symmetry group. If $S \in SO(4)$, then clearly

$$S^j_i S^k_j = \delta^k_i \tag{32}$$

implies that $SO(4)$ is a six parameter group. This can also be seen as the six rotation angles defined about the normal to the six unique Spatio-Temporal planes defined by any basis to $\mathbb{R}^4$. Ergo,

$$\{x^0, x^1\}, \{x^0, x^2\}, \{x^0, x^3\} = Spacio - Temporal \tag{33}$$
$$\{x^1, x^2\}, \{x^1, x^3\}, \{x^2, x^3\} = Spacial. \tag{34}$$

Furthermore to specify the angles of rotation about the normal to these planes we define the following rotation parameters

$$k_{01}, k_{02}, k_{03}, \theta_{12}, \theta_{13}, \theta_{23} \tag{35}$$

where the order of the indices denotes the orientation of the planes defining the rotation. We will see later that the infinitesimal matrices corresponding to these parameters are anti-symmetric under swapping the orientation of the defining planes. The rotation matrices defined by these planes are trivially represented in the canonical basis and are given by

$$R(k_{01}) = \begin{bmatrix} \cos k_{01} & -\sin k_{01} & 0 & 0 \\ \sin k_{01} & \cos k_{01} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R(k_{02}) = \begin{bmatrix} \cos k_{02} & 0 & -\sin k_{02} & 0 \\ 0 & 1 & 0 & 0 \\ \sin k_{02} & 0 & \cos k_{02} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{36}$$

$$R(k3) = \begin{bmatrix} \cos k_{03} & 0 & 0 & -\sin k_{03} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin k_{03} & 0 & 0 & \cos k_{03} \end{bmatrix} \tag{37}$$

$$R(\theta_{12}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{12} & -\sin\theta_{12} & 0 \\ 0 & \sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R(\theta_{13}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{13} & 0 & -\sin\theta_{12} \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_{13} & 0 & \cos\theta_{13} \end{bmatrix} \tag{38}$$

$$R(\theta_{23}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_{23} & -\sin\theta_{23} \\ 0 & 0 & \sin\theta_{23} & \cos\theta_{23} \end{bmatrix}. \tag{39}$$

The infinitesimal generators of $SO(4)$ are given by

$$\kappa_{ij} = \frac{\partial R(k_{ij})}{\partial k_{ij}}|_{k_{ij}=0} \tag{40}$$

$$\Theta_{ij} = \frac{\partial R(\theta_{ij})}{\partial \theta_{ij}}|_{\theta_{ij}=0}. \tag{41}$$

Ergo,

$$\kappa_{01} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \kappa_{02} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

$$\kappa_{03} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{43}$$

$$\Theta_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \Theta_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{44}$$

$$\Theta_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{45}$$

It is worth observing at this point that $\kappa_{ij} = -\kappa_{ji}$ and $\Theta_{ij} = -\Theta_{ji}$.

Now that we have the infinitesimal generators of our group we can begin building the irreducible representations of $SO(4)$ which will serve as our building blocks for the irreducible representations of $SO(3,1)^{\dagger}$. Consider the following generator matrices

$$B_1 = \frac{1}{2}(\Theta_{23} + \kappa_{01}),\ B_2 = \frac{1}{2}(\Theta_{31} + \kappa_{02}) \tag{46}$$

$$B_3 = \frac{1}{2}(\Theta_{12} + \kappa_{03}) \tag{47}$$

$$C_1 = \frac{1}{2}(\Theta_{23} - \kappa_{01}),\ C_2 = \frac{1}{2}(\Theta_{31} - \kappa_{02}) \tag{48}$$

$$C_3 = \frac{1}{2}(\Theta_{12} - \kappa_{03}), \tag{49}$$

whose commutators are given as

$$[B_i, C_j] = 0 \tag{50}$$

$$[B_i, B_j] = \epsilon_{ijk} B_k \tag{51}$$

$$[C_i, C_j] = \epsilon_{ijk} C_k. \tag{52}$$

Hence $SO(4)$ is isomorphic to the direct product group of two irreducible representations of $SO(3)$ which can be realized as

$$\begin{aligned} B_i &= A_i^{(j)} \otimes I_{2j'+1} && (53) \\ C_i &= I_{2j+1} \otimes A_i^{(j')}. && (54) \end{aligned}$$

Where the I_{2j+1} and $I_{2j'+1}$ are the identity matrices on $\mathbb{R}^{2j+1}$ and $\mathbb{R}^{2j'+1}$ respectively, with $A_i^{(j)}$ and $A_i^{(j')}$ the irreducible representations of $SO(3)$. Hence, $j, j' \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}$. Ergo, B_i and C_i are both of dimension $(2j+1)(2j'+1)$. Moreover, since the irreducible representations of product groups are equal to the direct product of the irreducible representations of each of the groups in the product we have that the irreducible representations of $SO(4)$ can be built from direct products of the irreducible representations of $SO(3)$.

The Representation Theory of $SO(3,1)^{\uparrow}$

We observed in the previous section the intimate connection between the symmetry groups $SO(3)$ and $SO(4)$. In this section we shall see how the irreducible representations of the Proper Orthochronous Lorentz group, $SO(3,1)^{\uparrow}$, can be generated from those of $SO(4)$ and hence the familiar $SO(3)$. We first note the obvious similarities between the $SO(4)$ and $SO(3,1)^{\uparrow}$.

Firstly we note that the Proper Orthochronous Lorentz group can be parameterized by 3 boost parameters and 3 Euler angles. Ergo, $SO(4)$ and $SO(3,1)^{\uparrow}$ are characterized by the same number of parameters. Moreover, given the signature we have chosen for the Minkowski metric $SO(4)$ and $SO(3,1)^{\uparrow}$ are symmetry groups of very similar quadratic forms given by

$$ds^2 = \delta_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{55}$$

and

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \tag{56}$$

respectively. In fact the similarity in their respective quadratic forms hints at the connection between the two and suggests that we recall the old *ict* formalism of the by-gone days of SR. Consider a rotation in the Spatio-Temporal plane $\{x^0, x^1\}$. Ergo,

$$\begin{bmatrix} x^{\bar{0}} \\ x^{\bar{1}} \end{bmatrix} = \begin{bmatrix} \cos k_{01} & -\sin k_{01} \\ \sin k_{01} & \cos k_{01} \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \tag{57}$$

which yields the set of equations

$$\begin{aligned} x^{\bar{0}} &= x^0 \cos k_{01} - x^1 \sin k_{01} && (58) \\ x^{\bar{1}} &= x^0 \sin k_{01} + x^1 \cos k_{01}. && (59) \end{aligned}$$

Recalling the old *ict* prescription let us transform coordinates and parameters as follows

$$\begin{aligned} x^0 &\longrightarrow ix^0 && (60) \\ x^{\bar{0}} &\longrightarrow ix^{\bar{0}} && (61) \\ k_{01} &\longrightarrow ik_{01}. && (62) \end{aligned}$$

Now equations (58) and (59) are given by

$$\begin{aligned} ix^{\bar{0}} &= ix^0 \cos ik_{01} - x^1 \sin ik_{01} && (63) \\ ix^{\bar{1}} &= ix^0 \sin ik_{01} + x^1 \cos ik_{01}, && (64) \end{aligned}$$

which reduce to

$$x^{\overline{0}} = x^0 \cosh k_{01} + x^1 \sinh k_{01} \tag{65}$$

$$x^{\overline{1}} = x^0 \sinh k_{01} + x^1 \cosh k_{01}. \tag{66}$$

What we have managed to accomplish is to transform the quadratic form with symmetry group $SO\left(4\right)$ into the quadratic form induced by the Minkowski metric with symmetry group $SO\left(3,1\right)^{\uparrow}$ while at the same time recovering the traditional form of the Lorentz transformation. Ergo, we have achieved Spatio-Temporal mixing whilst maintaining the spacial rotations. It is important to note at this point the difference between these groups and their parameters. As is obvious the boost parameters differ from the angular parameters of $SO\left(4\right)$ in that the boost parameters form an unbounded subset of $\mathbb{R}$ whilst the angular parameters are elements of a compact subset of $\mathbb{R}$.Thus $SO\left(3,1\right)^{\uparrow}$ is a non-compact group that differs from the compact $SO\left(4\right)$ in a number of ways. Firstly, in the unbounded parameters $SO\left(3,1\right)^{\uparrow}$ takes on unbounded non-periodic functions as opposed to the periodic sines and cosines of $SO\left(4\right)$. Moreover, the finite dimensional representations of $SO\left(3,1\right)^{\uparrow}$ are not unitary as we will see later. Lastly because $SO\left(3,1\right)^{\uparrow}$ is non-compact and hence an unbounded submanifold of its embedding space we have that the relationship between $SO\left(4\right)$ and $SO\left(3,1\right)^{\uparrow}$ holds only in a neighborhood of the identity map.

To obtain the irreducible representations of $SO\left(3,1\right)^{\uparrow}$ we define the following transformation matrix

$$T = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{67}$$

which carries $\left(x^0, x^1, x^2, x^3\right)$ into the desired $\left(ix^0, x^1, x^2, x^3\right)$. Thus, if $R \in SO\left(4\right)$ then the infinitesimal $R\left(k_{01}, k_{02}, k_{03}, \theta_{12}, \theta_{13}, \theta_{23}\right)$ is carried to $\Lambda \in SO\left(3,1\right)^{\uparrow}$ by the transformation

$$\Lambda = T^{-1} R\left(ik_{01}, ik_{02}, ik_{03}, \theta_{12}, \theta_{13}, \theta_{23}\right) T. \tag{68}$$

Furthermore, the infinitesimal generators of $SO\left(3,1\right)^{\uparrow}$ may be obtained from those of $SO\left(4\right)$ by the similarity transform defined above. Thus, following the notation of Jackson[2] we have

$$K_i = iT^{-1} \kappa_{0i} T \tag{69}$$

$$S_1 = T^{-1} \Theta_{23} T \tag{70}$$

$$S_2 = T^{-1} \Theta_{13} T \tag{71}$$

$$S_3 = T^{-1} \Theta_{12} T, \tag{72}$$

which obey commutation relations

$$\left[S_i, S_j\right] = \in_{ijk} S_k \tag{73}$$

$$\left[S_i, K_j\right] = \in_{ijk} K_k \tag{74}$$

$$\left[K_i, K_j\right] = - \in_{ijk} S_k. \tag{75}$$

We should note at this point since similarity transforms of this type leave commutators invariant these are the same commutators as $\{i\kappa_{01}, i\kappa_{02}, i\kappa_{03}, \Theta_{12}, \Theta_{13}, \Theta_{23}\}$. Furthermore, following Jackson[2] we may construct a general Lorentz transformation as

$$\Lambda = e^{-i\left(\boldsymbol{\omega} \cdot \mathbf{S} + \widehat{\beta} \cdot \mathbf{K} \tanh^{-1} \beta\right)}. \tag{76}$$

Finally we may derive the irreducible representations of $SO\left(3,1\right)^{\uparrow}$. We define the following infinitesimal matrices

$$B_i^{(+)} = \left(-1\right)^{i+1} S_i \tag{77}$$

$$B_i^{(-)} = K_i. \tag{78}$$

Therefore, the irreducible representations of $SO(3,1)^{\uparrow}$ are given by

$$\begin{aligned} B_i^{(+)} &= A_i^{(j)} \otimes I_{2j'+1} + I_{2j+1} \otimes A_i^{(j')} &(79)\\ B_i^{(-)} &= iA_i^{(j)} \otimes I_{2j'+1} - iI_{2j+1} \otimes A_i^{(j')}. &(80)\end{aligned}$$

References:
[1] Spinors In Physics, Jean Hladik ISBN: 0-387-98647-2
[2] Classical Electrodynamics, John D. Jackson ISBN: 0-471-30932-X