UNIVERSITY OF CALIFORNIA, SANTA BARBARA
Department of Physics

Physics 100A | Summer 2004
Prof. Jean Carlson

ASSIGNMENT 4

Part A due Wednesday, July 14. Part B due Monday, July 19.

The midterm will cover material up through and including Part A of the homework, which is due at 5:00 on Wednesday. Solutions will be posted then, and no late homework will be accepted. Part A includes some review problems for the midterm (the first four problems are last year's midterm!), as well as problems from material we cover in class this week that will be included on the midterm.

Reminder: Midterm exam in class on Thursday, July 15. The exam will be open book (Boas) and open notes, and will cover material through the end of the Spectral Theorem for Normal Operators, i.e. through the end of the Linear Algebra (Section A of the syllabus).

Keep reading Chapter 10 of Boas, and the book chapters on the Physics 100A course web site (http://www.eres.library.edu under Physics 100A).

DUE WEDNESDAY, JULY 14:

(1). Complex numbers $a + ib$, with a and b real, may be represented equivalently by a subset of the 2×2 matricies, which have the following form:

$$(a+ib) \Leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

(a). Show that this matrix representation has the same properties as complex numbers for (i) addition, i.e.,

$$(a+ib)+(c+id) \Leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

and (ii) multiplication, i.e.,

$$(a+ib)(c+id) \Leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

(b). Find the matrix corresponding to $(a+ib)^{-1}$.

(2). A and B are two noncommuting Hermitian matricies. Define the matrix C by

$$iC = [A, B] = AB - BA$$

(a). Prove C is Hermitian.

(b). Suppose a matrix M commutes with A and B. Prove that M also commutes with C

(3). The same similarity transformation diagonalizes each of two matricies A and B (A and B are not necessarily Hermitian). That is, $D_A = R^{-1}AR$, and $D_B = R^{-1}BR$, where D_A and D_B are diagonal matricies, and the same matrix R appears in both equations. Show the original matricies A and B must commute. (You may use without proof the fact that the product of two diagonal matricies is a diagonal matrix $D_A D_B = D_C$, in which the diagonal elements c_{ii} of the product matrix D_C are given by the the product of the diagonal elements a_{ii} of D_A and b_{ii} of D_B, i.e. $c_{ii} = a_{ii}b_{ii}$.)

(4). You are given a pair of equations

$$A|f_j>= \lambda_j|g_j>$$

$$A^T|g_j>= \lambda_j|f_j>$$

where A is a real, nonsingular $n \times n$ matrix, and A^T is the transpose of A. The sets of vectors $\{|f_j>\}$ and corresponding $\{|g_j>\}$ each consist of n column vectors. Each column vector has n elements. The n values of $\{\lambda_j\}$ are all different from one another. You may take as the inner product the usual inner product for complex vectors.

(a). Prove that $|f_j>$ is an eigenvector of (A^TA) with eigenvalue ${\lambda_j}^2$, and that $|g_j>$ is an eigenvector or (AA^T) with eigenvalue ${\lambda_j}^2$.

(b). State why each of the following statements are true:

(1) the $\{|f_j>\}$ form an orthogonal set

(2) the $\{|g_j>\}$ form an orthogonal set

(3) the $\{{\lambda_j}^2\}$ are real

(c). Prove that A can be written as

$$\hat{A} = \sum_{j=1}^{n} \lambda_j|g_j>< f_j|$$

with the $\{|f_j>\}$ and the $\{|g_j>\}$ normalized to unity.

Do the following problem from Shankar, QM:

(5). Exercise 1.8.5 (page 42): Consider the matrix

$$\Omega = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(a). Show that it is unitary.

(b). Show that its eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$.

(c). Find the corresponding eigenvectors and show that they are orthogonal.

(d). Verify that $U^\dagger \Omega U =$(diagonal matrix), where U is the matrix of eigenvectors of Ω.

Do the following problem from Shankar Basic Training:

(6). Problem 9.5.10 (page 265): Show that the following matricies commute and find a common basis of eigenvectors:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$N = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

PART B DUE MONDAY, JULY 19:

Part B takes you through two of the most fundamental applications of what you have learned so far in physics– normal modes of coupled oscillators and measurement in quantum mechanical spin systems. You don't need to have taken quantum or classical mechanics to do these problems, but if you have (or plan to) this will help you see how the mathematics of linear vector spaces fits in.

Read Section 9.6 (page 266) of Shankar Basic Training. This discusses applications of our results to a simple mass and spring system and works through the solution carefully. (Also see example 1.8.6 of Shankar QM page 46-54.) Then do the following problems from Basic Training:

(7). Problem 9.6.1 (page 272): Find the future state of the system if $x_1(0) = 1$ and $x_2(0) = 0$ at $t = 0$.

(8). Problem 9.6.2 (page 272): Find the normal modes and eigenfrequencies has a force constant $2k$.

The final problem relates the results of this class so far to fundamental results in quantum mechanics. Shankar has defined what you need to know about quantum mechanics in the problem. You can skip part (h) of the problem as it appears in Shankar (which repeats the sequence for a second matrix).

From Shankar Basic Training:

(9). Problem 9.5.11 (page 265): Consider the three spin-1 matricies

$$S_x = 2^{-1/2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_y = 2^{-1/2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which represent the components of the internal angular momentum of some elementary particle at rest (i.e. not related to the center of mass motion of the particle, but related to some motion internal to the particle). The operators $S^2 = {S_x}^2 + {S_y}^2 + {S_z}^2$ represents the total angular momentum squared. The dynamical state of the system is given by a vector (called the state vector) in the complex three dimensional space (i.e. V is the space of complex 3-tuples, and the state vector is in V) on which these matricies act. This simply means that all available information about the particle is stored in this vector.

According to the laws of Quantum Mechanics (i.e. take as given):

• A measurement of the angular momentum along any direction x, y or z will give only one of the eigenvalues of the corresponding spin operator. It can't have any other value.

• The probability that a given eigenvalue will result is equal to the absolute value squared of the inner product of the state vector with the corresponding eigenvector. (The state vector and all of the eigenvectors are normalized, so that the inner product of each with itself is equal to one. The is important if we want to have the interpretation of inner products as probabilities.)

• The state of the system immediately following this measurement will be the eigenvector (in this sense the process of measurement corresponds to a projection onto one of the eigenstates of the system).

(a). What are the possible values we can get if we measure the spin along the z-axis (i.e. what are the eigenvalues of the S_z matrix)?

(b). What are the possible values we can get if we measure the spin along the x- or y-axis?

(c). Say you made a measurement of S_x and got the largest possible value of S_x. What is the state vector immediately afterwards (i.e. what is the eigenvector of S_x corresponding to the largest eigenvalue)?

(d). If S_z is now measured (just after measuring S_x and getting the largest possible value), what are the odds of the various outcomes in the measurement of S_z? If you get the largest value of the possibilities, what is the state just after the measurement of S_z? Now, after measuring S_z, if you go back and measure S_x again, will you get the largest possible value again for S_x? Explain.

(e). What are the possible outcomes when S^2 is measured?

(f). From the four operators, S_x, S_y, S_z, and S^2, what is the largest number of simultaneously commuting operators you can pick at one time?

(g). A particle is in a state given by a column vector

$$|V>= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

First rescale the vector to normalize it. What are the odds for getting the three possible eigenvalues of S_z? What is the statistical or weighted average of these values? Compare this to $< V|S_z|V >$.