Practice Final Exam

Note From Prof M.: The rules stated below will also hold for the actual final exam. [Wednesday, March 20, 8-11 am]. Attempting the problems on this practice exam will give you a good idea of how prepared you are for the actual final exam. **Solutions are available on the course web site.** The questions below will give you some idea of what to expect on the actual final. However, the material covered on the actual final may be somewhat different. In particular, the material in problem 1 below overlaps with that on the midterm exam. There will be *at least* one such overlap problem on the actual final, but it may not resemble problem 1 below.

Office hours for the Final Exam: During Final Exam Week, Prof. Marolf's Office hours will be 3-5pm on Tuesday. If you need to make an additional appointment to speak with him, please contact him by e-mail at marolf@physics.ucsb.edu .

Instructions: You may use one (single-sided) page of notes that you bring with you. I recommend including formulae for Fourier and Laplace transforms. You may use any conventions your like (e.g., for where to place the factors of 2π), but be sure that you choose them consistently and that you understand them!

Except as noted above, this exam is closed book, closed notes, etc. No calculators are allowed. It is 2hrs and 45 minutes long. Show all of your work.

1. (15 points) Evaluate the following integral using contour integration (for real a with $a > 0$):

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2}.$$

2. (10 points) *Note: The following problem is two-dimensional.*

   Two conducting plates lie along the x and y axes, intersecting perpendicularly at the origin. The electric potential ϕ vanishes on both, but $\phi = 1$ at the point $(x, y) = (1, 1)$ Use the mapping function $w = z^2$ to find ϕ in the quadrant where $x, y > 0$.

3. (15 points) Using integration by parts, find the leading order term in the asymptotic series for

$$I(x) = \int_x^\infty \cos(u^2) du$$

in the limit where x is large, positive, and real valued. (Hint: write $\cos(u^2) = \mathrm{Re}(e^{iu^2})$., where Re denotes the real part.)

4. (20 points) Use Fourier transforms to solve the following equation for $y(x)$, where $-\infty < x < \infty$:

$$y'(x) - 4y(x) = \Theta(x)e^{-4x}.$$

Here $\Theta(x)$ is the Heaviside step function given by

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}.$$

We seek a solution $y(x)$ satisfying $|y(x)| \to 0$ as $x \to \infty$.

5. (15 points)

Consider the differential equation

$$u''(x) + u(x) = \sin(2x).$$

Using Laplace transforms, solve this initial value problem for $u(x)$ with $u(0) = u'(0) = 0$.

6. (25 points: note that this problem has two parts.) Consider the following ordinary differential equation

$$\frac{d^2y(x)}{dx^2} = f(x),$$

where the forcing function $f(x)$ will be specified in part (b) below. We wish to solve this equation on the interval $x \in [0, 1]$ subject to the boundary conditions $y(0) = 0$ and $y'(1) = dy/dx|_{x=1} = 0$.

a) Find the Green's function $G(x, x')$ which satisfies

$$\frac{d^2G(x, x')}{dx^2} = \delta(x - x')$$

and the boundary conditions stated above.

b) Use the Green's function method and the results of part (a) to solve for $y(x)$ when $f(x)$ is given by

$$\Theta(x) = \begin{cases} x, & 0 \leq x < 1/2 \\ 0, & 1/2 \leq x \leq 1 \end{cases}.$$