

**Note about Office hours for the Final Exam:** During Final Exam Week, Prof. Marolf's Office hours will be 3-5pm on Thursday. If you need to make an additional appointment to speak with him, please contact him by e-mail at marolf@physics.ucsb.edu .

## Sample Solutions to the Practice Final Exam<sup>1</sup>

1. (10 points) Evaluate the following integral using contour integration (for real  $a$  with  $a > 0$ ):

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}.$$

**Solution:**

$$\begin{aligned} I &= \int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x + ia)^2(x - ia)^2} \\ \int_{\Gamma} \frac{dz}{(z + ia)^2(z - ia)^2} &= 2\pi i \left[ \frac{d}{dz} \frac{1}{(z + ia)^2} \right]_{ia} \\ &= 2\pi i \frac{-2}{(2ia)^3} \\ &= \frac{\pi}{2a^3} \\ \int_{\Gamma} &= \int_{-\infty}^{\infty} + \int_{\Gamma_R} \\ \lim_{R \rightarrow \infty} \int_{\Gamma_R} &\rightarrow 0 \\ \Rightarrow \int_{\Gamma} &= 2I \\ \Rightarrow I &= \frac{\pi}{4a^3} \end{aligned}$$

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<sup>1</sup>Based on solutions by Michael Gary from Winter 2007.

2. (10 points) *Note: The following problem is two-dimensional.*

Two conducting plates lie along the  $x$  and  $y$  axes, intersecting perpendicularly at the origin. The electric potential  $\phi$  vanishes on both, but  $\phi = 1$  at the point  $(x, y) = (1, 1)$ . Use the mapping function  $w = z^2$  to find  $\phi$  in the quadrant where  $x, y > 0$ .

**Solution:** Consider the action of the mapping function  $w = f(z) = z^2$  on the two plates. One plate occupies the points where  $z = x + iy$  is real while the other occupies the points where  $z = x - iy$  is imaginary. For both plates,  $w = z^2$  is real. So, the map  $w = z^2$  takes both plates to the real  $w$ -axis. Since  $\phi = 0$  on the both plates, our boundary conditions will be satisfied if in the  $w$ -plane we find a potential function such that  $\phi(w) = 0$  when  $w$  is real.

We also need  $\phi = 1$  when  $z = 1 + i$ ; i.e., when  $w^2 = (1 + i)^2 = 1 + 2i - 1 = 2i$ .

In the  $w$ -plane it is easy to solve Laplace's equation subject to the conditions above. As usual, we first try to make  $\phi$  linear. Writing  $w = u + iv$ , we try a solution of the form  $\phi = Av + B$  where  $A$  and  $B$  are both real. We want

i)  $\phi = 0$  when  $v = 0$ . This sets  $B = 0$ .

ii)  $\phi = 1$  when  $(u, v) = (0, 2)$ . Thus,  $2A = 1$  and  $A = 1/2$ .

Having found that  $\phi = v/2 = \text{Im } w/2$ , we now substitute  $w = z^2$  to find

$$\phi(z) = \text{Im } (z^2/2) = \text{Im } ((x^2 + 2ixy - y^2)/2) = xy.$$

This is the solution in the quadrant where  $x, y > 0$ .

*Note: It is also a valid solution in the other quadrants. However, in this physics problem the Laplace equation does not need to hold at the plates themselves (because there can be surface charges on the plates). So, it would not be physically meaningful to use the condition on  $\phi$  at  $(x, y) = (1, 1)$  to determine the solution in the other quadrants.*

3. (10 points) Using integration by parts, find the leading order term in the asymptotic series for

$$I(x) = \int_x^\infty \cos(u^2) du$$

in the limit where  $x$  is large, positive, and real valued. (Hint: write  $\cos(u^2) = \operatorname{Re}(e^{iu^2})$ , where  $\operatorname{Re}$  denotes the real part.)

**Solution:**

$$\begin{aligned} I(x) &= \int_x^\infty \cos(u^2) du \\ &= \int_x^\infty \frac{1}{2u} \frac{d}{du} \sin(u^2) du \\ &= \frac{\sin(u^2)}{2u} \Big|_x^\infty + \int_x^\infty \frac{\sin(u^2)}{2u^2} du \\ &= -\frac{\sin(x^2)}{2x} + \dots \end{aligned}$$

4. (20 points) Use Fourier transforms to solve the following equation for  $y(x)$ , where  $-\infty < x < \infty$ :

$$y'(x) - 4y(x) = \Theta(x)e^{-4x}.$$

Here  $\Theta(x)$  is the Heaviside step function given by

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

We seek a solution  $y(x)$  satisfying  $|y(x)| \rightarrow 0$  as  $x \rightarrow \infty$ .

**Solution:** We will work in the standard quantum conventions, where

the  $1/2\pi$  is placed on the inverse Fourier transform.

$$\begin{aligned}
 y'(x) - 4y(x) &= \theta(x)e^{-4x} \\
 \Rightarrow ik\tilde{y}(k) - 4\tilde{y}(k) &= \int_0^\infty e^{-(4+ik)x} dx \\
 &= \frac{1}{4+ik} \\
 \Rightarrow \tilde{y}(k) &= \frac{-1}{(4+ik)(4-ik)} \\
 \Rightarrow y(x) &= \frac{-1}{2\pi} \int_{-\infty}^\infty \frac{e^{ikx}}{(k-4i)(k+4i)} dk \\
 &= -i\theta(x) \frac{e^{ix(4i)}}{8i} + i\theta(-x) \frac{e^{i(-4i)x}}{-8i} \\
 &= -(\theta(x) + \theta(-x))e^{-4|x|}/8 = -e^{-4|x|}/8
 \end{aligned}$$

Note that if  $x > 0$ , we need to close the contour around the upper half plane, picking up the pole at  $k = 4i$ , but if  $x < 0$  we need to close around the lower half plane, picking up the pole at  $k = -4i$ . For the  $x < 0$  term we must remember to include the  $(-)$  sign from integrating clockwise around the pole.

5. (15 points)

Consider the differential equation

$$u''(x) + u(x) = \sin(2x).$$

Using Laplace transforms, solve this initial value problem for  $u(x)$  with  $u(0) = u'(0) = 0$ .

**Solution:**

$$\begin{aligned}
u''(x) + u(x) &= \sin(2x) \\
\Rightarrow -u'(0) - su(0) + s^2U(s) + U(s) &= \frac{2}{s^2 + 4} \\
\Rightarrow (s^2 + 1)U(s) &= \frac{2}{(s + 2i)(s - 2i)} \\
\Rightarrow U(s) &= \frac{2}{(s + i)(s - i)(s + 2i)(s - 2i)} \\
\Rightarrow u(x) &= \frac{-\sin(2x)}{3} + \frac{2\sin(x)}{3} \\
\Rightarrow u''(x) &= \frac{4}{3}\sin(2x) - \frac{2}{3}\sin(x) = \phi(x)
\end{aligned}$$

6. (20 points: note that this problem has two parts.) Consider the following ordinary differential equation

$$\frac{d^2y(x)}{dx^2} = f(x),$$

where the forcing function  $f(x)$  will be specified in part (b) below. We wish to solve this equation on the interval  $x \in [0, 1]$  subject to the boundary conditions  $y(0) = 0$  and  $y'(1) = dy/dx|_{x=1} = 0$ .

- a) Find the Green's function  $G(x, x')$  which satisfies

$$\frac{d^2G(x, x')}{dx^2} = \delta(x - x')$$

and the boundary conditions stated above.

- b) Use the Green's function method and the results of part (a) to solve for  $y(x)$  when  $f(x)$  is given by

$$\Theta(x) = \begin{cases} x, & 0 \leq x < 1/2 \\ 0, & 1/2 \leq x \leq 1 \end{cases}$$

**Solution:**

We want a Green's function for  $\frac{d^2}{dx^2}$  with boundary conditions  $y(0) = 0, y'(1) = 0$ . The equation  $y''(0) = 0$  has solution  $a + bx$ . Thus,

$$G(x, x') = \begin{cases} a_1 + b_1x & 0 < x < x' \\ a_2 + b_2x & x' < x < 1 \end{cases} \quad (1)$$

Since  $y(0) = 0$ , we know that  $a_1 = 0$ . Similarly,  $y'(1) = 0$  means that  $b_2 = 0$ .

$$\begin{aligned} \frac{d^2 G(x, x')}{dx^2} &= \delta(x - x') \\ \Rightarrow \left[ \frac{dG(x, x')}{dx} \right]_{x'-\epsilon}^{x'+\epsilon} &= 1 \\ \Rightarrow 0 - b_1 &= 1 \\ \Rightarrow G(x, x') &= \begin{cases} -x & 0 < x < x' \\ -x' & 0 < x' < x \end{cases} \end{aligned}$$

$$\begin{aligned} y(x) &= \int_0^1 G(x, x') f(x') dx' \\ &= \int_0^{1/2} G(x, x') x' dx' \\ &= \left( \int_0^x (-x) x' dx' + \int_x^{1/2} (-x') x' dx' \right) \\ &= (-x^3/2 + x^3/3 - 1/24) \\ &= (-1/24 - x^3/6) \end{aligned}$$