Physics 101 Mid-Term Exam
2/11/13
Complex Analysis

Your Name: Jason Kaufman____________ **Your seat #:** ____________

Instructions: You may use one (single-sided) page of notes that you have brought with you (and which you must turn in with and stable to your exam). The use of books, calculators, additional notes beyond those mentioned above, or other aids will not be allowed. The exam is 50 minutes long. Show all of your work. Extra paper is available if you need some. **Remember to write your name at the top of the exam.**

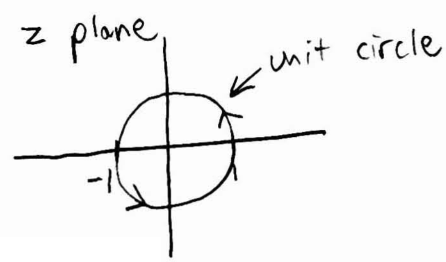
1. (30 points) Evaluate

$$PV \int_0^{2\pi} \frac{d\theta}{1+e^{-i\theta}}$$

using Cauchy's principal value prescription.

NOTE: There are several ways to do this problem. The following is the fastest and easiest, and most likely what the professor expected.

$z = e^{i\theta}$, $dz = e^{i\theta} i\,d\theta = z\,i\,d\theta$



$$PV \int_C \frac{-i\,dz}{z(1+\frac{1}{z})} = PV \int_C \frac{-i\,dz}{z+1}$$

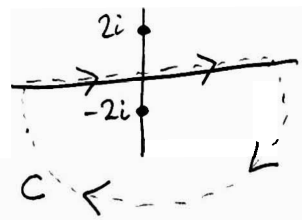
$$= 2\pi i \left(\tfrac{1}{2}\right) \Sigma \,\text{Res} = 2\pi i \left(\tfrac{1}{2}\right)(-i) = \boxed{\pi}$$

2. (30 points) Evaluate

$$I = \int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2+4} dx$$

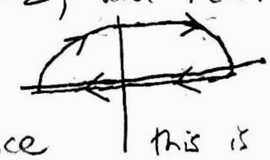
by contour integration.

z plane



Consider $I_2 = \int_C \frac{ze^{-iz}}{z^2+4} dz$

By Jordan's lemma, the semi-circle contribution to the integral goes to zero in the limit that the radius goes to infinity.
(This is intuitive since $e^{-iz} = e^{-i(z_1+iz_2)} = e^{-iz_1+z_2}$ In the lower half plane, the real part of $-iz_1+z_2$ is negative, making the exponential term vanishingly small as the radius goes to infinity. More rigorously, let $w=-z$, and rewrite the integral as $\int_{C'} \frac{we^{iw}}{w^2+4} dw$, where C' is



By Jordan's lemma, the semicircle vanishes since this is of the form $\int e^{iaz} f(z) dz$ where $a>0$ and $f(z)\to 0$ on the semicircle.
Since this semi-circle maps to the semi-circle in the lower half plane in $\int_C \frac{ze^{iz}}{z^2+4} dz$, the semicircle in C also vanishes. On an exam you do NOT need to explain this. Just check that $Re(-iz_1+z_2)<0$ and quote Jordan's lemma).

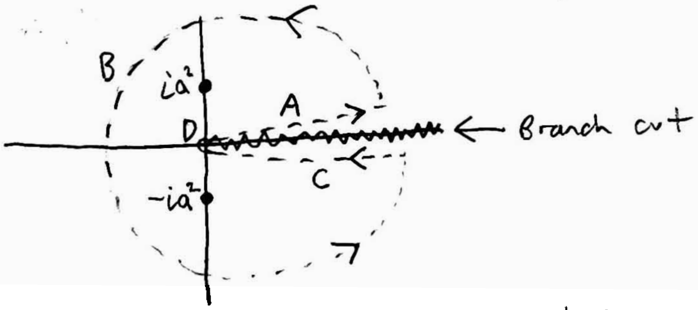
$$I = I_2 = -2\pi i \sum Res = (-2\pi i)\left(\frac{-2i\, e^{-i(-2i)}}{-4i}\right) = -\pi i e^{-2}$$

3. (30 points) Evaluate

$$I = \int_0^\infty \frac{x^{1/2}}{x^2+a^4}dx$$

by contour integration for arbitrary real positive a.

Consider $I_2 = \int_C \frac{z^{\frac{1}{2}}}{z^2+a^4}$, where the branch cut for $z^{\frac{1}{2}}$ is on the positive reals so that $z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{i\theta/2}$, $0<\theta<2\pi$, and the contour is



(NOTE: There are other ways to do this problem, but this is how similar problems were solved in class and in the HW solutions.)

$I_2 = A + B + C + D$

$$A = \lim_{\substack{C_A \to \text{positive} \\ \text{real axis}}} \int_{C_A} \frac{|z|^{\frac{1}{2}} e^{i\theta/2}}{z^2+a^4} dz = \int_0^\infty \frac{z^{\frac{1}{2}}}{z^2+a^4} dz = I$$

$$C = \lim_{\substack{C_C \to \text{positive} \\ \text{real axis}}} \int_{C_C} \frac{|z|^{\frac{1}{2}} e^{i\theta/2}}{z^2+a^4} dz = \int_\infty^0 \frac{z^{\frac{1}{2}} e^{i2\pi/2}}{z^2+a^4} dz = I$$

$B = 0$ since the highest power in the denominator is more than one greater than the highest power in the numerator. NOTE: You can prove this is sufficient by letting $z = re^{i\theta}$ and letting $r \to \infty$. NOTE: This is NOT Jordan's lemma!

$D = 0$ since the integral of something finite over a vanishingly small contour is zero.

$$I = \frac{1}{2}I_2 = \frac{1}{2}(2\pi i)\left(\frac{ae^{i\pi/4}}{2ia^2} + \frac{ae^{i3\pi/4}}{-2ia^2}\right) = \frac{\pi}{a} i e^{\pi i/2}\left(\frac{e^{-i\pi/4} - e^{i\pi/4}}{2i}\right) = -\frac{\pi}{a}(-\sin \pi/4)$$

$$= \boxed{\frac{\sqrt{2}}{2}\left(\frac{\pi}{a}\right)}$$

4. (10 points) Let $u = 2x^3 - 3x^2y - 6xy^2 + y^3$. Is this the real part of an analytic function $f = u + iv$ of $z = x + iy$? *Briefly* explain your reasoning. (This need not involve any sentences. A short calculation may be sufficient. But a "yes" or "no" alone is worth no credit.)

Find $\nabla^2 u$:

$\partial_x^2 u = 12x - 6y$

$\partial_y^2 u = -12x + 6y$

$\nabla^2 u = 0$ so u is the real part of an analytic function.

NOTE: You could instead find v with the Cauchy-Riemann equations, but this takes longer.