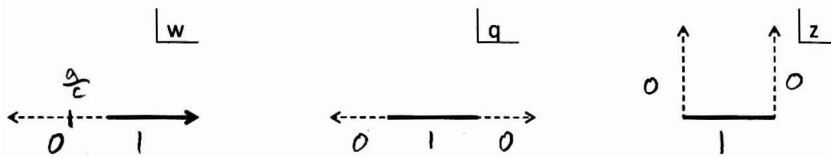


1. (50 points) We wish to solve the 2d Laplace equation for an electric potential Φ subject to the following three sets of boundary conditions. The solid lines indicate conductors where $\Phi = 0$, while the dashed lines indicate conductors where $\Phi = 1$. Segments with arrows should be regarded as stretching to infinity in the direction indicated by the arrows. The diagrams are drawn in the complex w , q , and z -planes as indicated. The finite horizontal segments (solid line segments, q and z diagrams only) cover the interval $[-\pi, \pi]$ on the real axis.



Show all of your work for each part below.

a) Show that a fractional linear transformation of the form $w = \frac{aq+b}{cq+d}$ relates the above boundary conditions in the w - and z -planes. (Remember to check the line segments and not just the corners.) Find a, b, c, d .

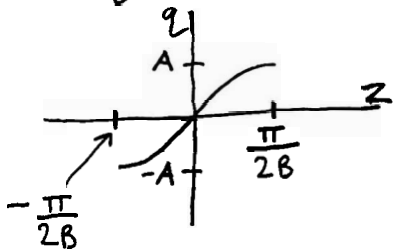
b) Show that a transformation of the form $q = A \sin Bz$ relates the above boundary conditions in the q - and z -planes. Find A, B . (Remember to check the line segments and not just the corners.)

c) Recall that $\Phi = \frac{1}{\pi} \text{Im}(\ln w)$ solves Laplace's equation with the indicated boundary conditions in the w -plane. Use your solutions above to construct a solution to Laplace's equation subject to the boundary conditions indicated in the z -plane. It is sufficient to write your solution in terms of the complex coordinate z . You do not need to express the result in terms of either Cartesian or Polar coordinates in this plane, and you do not need to simplify your answer in any way.

There is not enough room to work the problem on this page, but I have provided extra paper below. Start on the next page.

a) This answer was given at the beginning of the exam: $w = \frac{q + \pi}{-q + \pi}$ NOTE: This is not the only answer, but it was given so we'll use it.

b) Plot $q = A \sin Bz$ for $-\pi \leq z \leq \pi$



Clearly if $A = \pi$ and $B = \frac{1}{2}$ then the solid line in z maps to the solid line in q .

What about the dashed lines?

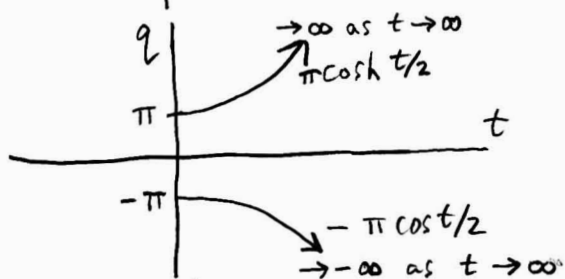
In z , we can parametrize them: $z = \pm\pi + it$, $t \geq 0$

$$q = \pi \sin \frac{z}{2} = \pi \left(\frac{e^{iz/2} - e^{-iz/2}}{2i} \right) = \frac{\pi}{2i} \left(e^{\frac{i}{2}(\pm\pi + it)} - e^{-\frac{i}{2}(\pm\pi + it)} \right)$$

$$= \frac{\pi}{2i} \left(e^{\pm \frac{\pi i}{2}} e^{-t/2} - e^{\mp \frac{\pi i}{2}} e^{t/2} \right) = \frac{\pi}{2i} \left(\pm i e^{-t/2} - (\mp i) e^{t/2} \right)$$

$$= \pm \pi \left(\frac{e^{t/2} + e^{-t/2}}{2} \right) = \pm \pi \cosh t/2$$

We can plot these for $t \geq 0$



So the dashed lines in z map to the dashed lines in q

c) $\Phi = \frac{1}{\pi} \text{Im} \left(\ln \left(\frac{\pi \sin \frac{z}{2} + \pi}{-\pi \sin \frac{z}{2} + \pi} \right) \right)$

②

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{iak} - (1+iak)}{k^2} e^{ikx} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ik(x+a)} - (1+iak)e^{ikx}}{k^2} dk$$

Not singular at $k=0$!

$$\lim_{k \rightarrow 0} \frac{e^{iak} - (1+iak)}{k^2} = \frac{1 + iak + (iak)^2 \dots - (1+iak)}{k^2} = (ia)^2$$

Since analytic near the real axis (it's analytic everywhere!), we can deform the contour without changing $f(x)$.

Choose contour C :

if we have to,

This way we can evaluate each term separately without worrying about each term's pole at $k=0$ (NOTE: Each term separately is singular at $k=0$, even though together they are non-singular).

For $x > 0$, close in UHP. $f(x) = 0$ by Cauchy's thm. } Keep terms together so if
 For $x < -a$, close in LHP. $f(x) = 0$ by Cauchy's thm. } there's no pole at $k=0$

For $-a < x < 0$:

$$\text{Let } f_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ik(x+a)}}{k^2} dk \quad \text{and} \quad f_2(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-(1+iak)}{k^2} e^{ikx} dk$$

$$f(x) = f_1(x) + f_2(x)$$

Close $f_1(x)$ in UHP and close $f_2(x)$ in LHP. $f_2(x) = 0$ by Cauchy's thm.

$$f_1(x) = \frac{2\pi i}{\sqrt{2\pi}} \left. \frac{d}{dk} (e^{ik(x+a)}) \right|_{k=0} = \frac{2\pi i}{\sqrt{2\pi}} i(x+a) = -\sqrt{2\pi} (x+a)$$

$$f(x) = \begin{cases} 0, & x < -a \text{ \& } x > 0 \\ -\sqrt{2\pi} (x+a), & -a < x < 0 \end{cases}$$

What about $x = -a$ and $x = 0$? (optional)

f satisfies the Dirichlet conditions (see Boas or class notes), so at discontinuities $f =$ the midpoint of the values on either side.

So, $f(x) = 0$ at $x = -a$ (Actually, $f(x)$ is continuous here) and

$$f(x) = \frac{1}{2} (-\sqrt{2\pi} a + 0) = -\frac{\sqrt{2\pi}}{2} a \quad \text{at } x = 0.$$

NOTE: This problem is nearly identical to the two Fourier inverse problems on HW8. See HW8 solutions for more details.