Physics 101 Homework 10 Solutions

Michael Gary, modified by Michael Johnson

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1 Ch. 11, §3, 5

$$\frac{\Gamma(1/2)\Gamma(4)}{\Gamma(9/2)} = \frac{\Gamma(1/2)\Gamma(4)}{(7/2)(5/2)(3/2)(1/2)\Gamma(1/2)} = \frac{2^5 3!}{7(5)3} = 32/35 \tag{1.1}$$

2 Ch. 11, §3, 11

$$\int_0^\infty x^5 e^{-x^2} dx = \frac{1}{2}\int_0^\infty u^2 e^{-u} du = \Gamma(3)/2 = 1 \tag{2.1}$$

3 Ch. 11, §5, 5

$$\Gamma(1/2-n)\Gamma(1/2+n) = \Gamma((1/2+n))\Gamma(1-(1/2+n)) = \frac{\pi}{\sin(\pi(n+1/2))} = (-1)^n \pi \tag{3.1}$$

$$z!(-z)! = \Gamma(z+1)\Gamma(1-z) = z\Gamma(z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)} \tag{3.2}$$

4 Ch. 11, §5, 6

$$\frac{d^n}{dp^n}\Gamma(p) = \frac{d^n}{dp^n}\int_0^\infty x^{p-1}e^{-x}dx \tag{4.1}$$

$$= \int_0^\infty \frac{d^n}{dp^n} e^{(p-1)\log x} e^{-x} dx \tag{4.2}$$

$$= \int_0^\infty (\log x)^n e^{(p-1)\log x} e^{-x} dx \tag{4.3}$$

$$= \int_0^\infty x^{p-1} e^{-x} (\log x)^n dx \tag{4.4}$$

5 Ch. 11, §6, 1

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = -\int_1^0 (1-y)^{p-1} y^{q-1} dy = B(q,p) \tag{5.1}$$

6 Ch. 11, §10, 2

$$\begin{aligned}
\Gamma(p,x) &= \int_x^\infty t^{p-1}e^{-t}dt && (6.1)\\
&= \left[-t^{p-1}e^{-t}\right]_x^\infty + (p-1)\int_x^\infty t^{p-2}e^{-t}dt && (6.2)\\
&= x^{p-1}e^{-x} + (p-1)x^{p-2}e^{-x} + (p-1)(p-2)x^{p-3}e^{-x} + \cdots && (6.3)
\end{aligned}$$

7 Ch. 11, §11, 5

$$\begin{aligned}
\sqrt{n}\Gamma(n+1) &\sim \sqrt{n}\sqrt{2\pi n}n^n e^{-n} = \sqrt{2\pi}n^{n+1}e^{-n} && (7.1)\\
\Gamma(n+3/2) &\sim \sqrt{2\pi(n+1/2)}(n+1/2)^{n+1/2}e^{-n-1/2} && (7.2)\\
&= \sqrt{2\pi}(n+1/2)^{n+1}e^{-n-1/2} && (7.3)\\
\Rightarrow \lim_{n\to\infty}\frac{\Gamma(n+3/2)}{\sqrt{n}\Gamma(n+1)} &= \lim_{n\to\infty}\frac{\sqrt{2\pi}(n+1/2)^{n+1}e^{-n}e^{-1/2}}{\sqrt{2\pi}n^{n+1}e^{-n}} && (7.4)\\
&= \lim_{n\to\infty} e^{-1/2}\frac{n^{n+1}\left((1+\frac{1}{2n})^{2n+2}\right)^{1/2}}{n^{n+1}} && (7.5)\\
&= e^{-1/2}e^{1/2} = 1 && (7.6)
\end{aligned}$$

8 Ch. 11, §11, 10

$$\begin{aligned}
\lim_{n\to\infty}\frac{(n!)^{1/n}}{n} &= \lim_{n\to\infty}\frac{(\sqrt{2\pi n}n^n e^{-n})^{1/n}}{n} && (8.1)\\
&= \lim_{n\to\infty}\frac{(2\pi n)^{1/2n}n}{ne} && (8.2)\\
&= \lim_{n\to\infty}\frac{e^{\log(n)/2n}}{e} = 1/e && (8.3)
\end{aligned}$$

9 Ch. 11, §13, 3

$$\begin{aligned}
\lim_{n\to\infty} n^x B(x,n) &= \lim_{n\to\infty}\frac{n^x\Gamma(x)\Gamma(n)}{\Gamma(n+x)} && (9.1)\\
&= \lim_{n\to\infty}\Gamma(x)n^x\frac{\sqrt{2\pi}(n-1)^{n-1}e^{-n+1}}{\sqrt{2\pi}(x+n-1)^{x+n-1}e^{-n-x+1}} && (9.2)\\
&= \lim_{n\to\infty}\Gamma(x)e^x n^x\frac{n^{n-1}(1-1/n)^{n-1}}{n^{n+x-1}(1+(x-1)/n)^{n+x-1}} && (9.3)\\
&= \lim_{n\to\infty}\Gamma(x)\frac{n^x n^{n-1}e^x e^{-1}}{n^{n+x-1}e^{x-1}} = \Gamma(x) && (9.4)
\end{aligned}$$

10 Ch. 11, §13, 4

$$\int_0^\infty \frac{e^{-t}dt}{1+xt} = \left[\frac{-e^{-t}}{1+xt}\right]_0^\infty - \int_0^\infty \frac{xe^{-t}dt}{(1+xt)^2} \quad (10.1)$$

$$= 1 + \left[\frac{xe^{-t}}{(1+xt)^2}\right]_0^\infty + \int_0^\infty \frac{2x^2e^{-t}dt}{(1+xt)^3} \quad (10.2)$$

$$= 1 - x + \left[\frac{2x^2e^{-t}}{(1+xt)^3}\right]_0^\infty + \int_0^\infty \frac{6x^3e^{-t}dt}{(1+xt)^4} \quad (10.3)$$

$$= \sum(-1)^n n! x^n \quad (10.4)$$