Physics 101 Homework 2 Solutions

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January 18, 2013

1 Ch. 2, §17, p. 81, 19

We want to show $\tan^{-1} z = \frac{1}{2i}\log\frac{1+iz}{1-iz}$. Let

$$\begin{aligned} w &= \frac{1}{2i}\log\frac{1+iz}{1-iz} && (1.1)\\ \Rightarrow 2iw = \log\frac{1+iz}{1-iz} & && (1.2)\\ \Rightarrow \frac{1+iz}{1-iz} &= e^{2iw} && (1.3)\\ \Rightarrow 1+iz &= e^{2iw} - ize^{2iw} && (1.4)\\ \Rightarrow iz(1+e^{2iw}) &= e^{2iw} - 1 && (1.5)\\ \Rightarrow z &= -i\frac{e^{2iw}-1}{e^{2iw}+1} && (1.6)\\ &= -i\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} && (1.7)\\ &= \tan w && (1.8) \end{aligned}$$

2 Ch. 2, §17, p. 81, 25

2.1 (a) $\overline{\cos z} \stackrel{?}{=} \cos\overline{z}$

$$\overline{\cos z} = \overline{\frac{1}{2}(e^{iz}+e^{-iz})} = \frac{1}{2}(e^{-i\overline{z}}+e^{i\overline{z}}) = \cos\overline{z} \tag{2.1}$$

2.2 (b) $\overline{\sin z} \stackrel{?}{=} \sin\overline{z}$

$$\overline{\sin z} = \overline{\left(\frac{e^{iz}-e^{-iz}}{2i}\right)} = \frac{e^{-i\overline{z}}-e^{i\overline{z}}}{-2i} = \frac{e^{i\overline{z}}-e^{-i\overline{z}}}{2i} = \sin\overline{z} \tag{2.2}$$

2.3 (c) $\overline{1+iz} \stackrel{?}{=} 1+i\overline{z}$

$$\overline{1+iz} = 1-i\overline{z} \neq 1+i\overline{z} \tag{2.3}$$

2.4 (d)

$$\overline{f(z)} = \overline{\sum_{n=0}^{\infty} a_n z^n} = \sum_{n=0}^{\infty} \overline{a}_n \overline{z}^n = \sum_{n=0}^{\infty} a_n \overline{z}^n = f(\overline{z}) \tag{2.4}$$

2.5 (e)

Note that the series expansion of $\sinh z$ has only real coefficients. Therefore, $\overline{\sinh z} = \sinh \overline{z}$. Since the imaginary part of $\sinh(z)$ is $\left(\frac{-i}{2}\right)\left(\sinh(z) - \overline{\sinh(z)}\right) = \frac{1}{2}\left(-i\sinh(z) + i\sinh(\overline{z})\right)$, $-i\sinh(z) + i\sinh(\overline{z})$ is purely real. In this case, we simply have $z = 1 - i$.

3 Branch Cut Problems

3.1

I think it's helpful to first assign unique polar coordinates to points in the domain. Let's choose $-3\pi/2 < \theta < \pi/2$. (Of course, we could add any integer multiple of 2π to the bounds, so that this is indeed a choice.) Now write the multivalued function and solve for the correct branch given the constraint $f(1) = 1$:

$$\begin{aligned} f(z) = z^{\pi} &= r^{\pi} e^{\pi(i\theta + 2\pi i N)} \\ 1 = f(1) &= e^{\pi(0 + 2\pi i N)} \end{aligned}$$

so $N = 0$ and $f(z) = r^{\pi} e^{\pi(i\theta)}$. So $f(-1) = e^{\pi i(-\pi)} = e^{-\pi^2 i}$.

3.2

This time let's choose $-\pi/2 < \theta < 3\pi/2$. Since the multivalued function and the constraint are the same, by identical reasoning we get $f(z) = r^{\pi} e^{\pi(i\theta)}$ as before. But this time $f(-1) = e^{\pi i(\pi)} = e^{\pi^2 i} \neq e^{-\pi^2 i}$.

3.3

Let $z + 1 = r_1 e^{i\theta_1}$ and $z - 1 = r_2 e^{i\theta_2}$. Choose the domain to be $-\pi < \theta_1, \theta_2 < \pi$. The multivalued function f_1 is $f_1 = r_1^{\frac{1}{2}} e^{\frac{1}{2}(i\theta_1 + 2\pi i N)}$. Since $f_1(2) = \sqrt{3}$, the correct branch is then $f_1 = r_1^{\frac{1}{2}} e^{\frac{1}{2} i\theta_1}$. The multivalued function f_2 is $f_2 = r_2^{\frac{1}{2}} e^{\frac{1}{2}(i\theta_2 + 2\pi i N)}$. Since $f_2(2) = 1$, the correct branch is then $f_2 = r_2^{\frac{1}{2}} e^{\frac{1}{2} i\theta_2}$. Now we can calculate the limits on both sides of the branch cuts.

$$\lim_{z \to -5 + i\epsilon} f_1(z) = \lim_{z \to (r_1 = 4, \theta_1 = \pi)} f_1(z) = 2e^{\frac{\pi}{2} i} = 2i$$

$$\lim_{z \to -5 - i\epsilon} f_1(z) = \lim_{z \to (r_1 = 4, \theta_1 = -\pi)} f_1(z) = 2e^{-\frac{\pi}{2} i} = -2i$$

$$\lim_{z \to -5+i\epsilon} f_2(z) = \lim_{z \to (r_2=6, \theta_2=\pi)} f_2(z) = \sqrt{6} e^{\frac{\pi}{2} i} = \sqrt{6} i$$

$$\lim_{z \to -5-i\epsilon} f_2(z) = \lim_{z \to (r_2=6, \theta_2=-\pi)} f_2(z) = \sqrt{6} e^{-\frac{\pi}{2} i} = -\sqrt{6} i$$

Consider defining the single valued function $f(z) = \sqrt{(z+1)(z-1)}$ as $f(z) = f_1(z) f_2(z)$ where the branches of f_1 and f_2 are the same as above, and the branch cut is then the union of the branch cuts above, i.e., the real numbers less than or equal to 1. Since f_1 and f_2 are continuous on the resulting domain, their product $f(z)$ is also continuous. Question: Can we remove the branch cut for real numbers less than -1? We can define $f(z)$ along the real numbers less than -1 as the limit as $f(z)$ approaches these values. If this limit is well-defined (which we will check), then $f(z)$ is continuous on this set, too, and so we can remove the branch cut.

$$\lim_{z \to x+i\epsilon, x<-1} f(z) = \lim_{\theta_1, \theta_2 \to \pi} \sqrt{r_1 r_2} e^{\frac{1}{2} i(\theta_1+\theta_2)} = \sqrt{r_1 r_2} e^{\pi i} = -\sqrt{r_1 r_2}$$

$$\lim_{z \to x-i\epsilon, x<-1} f(z) = \lim_{\theta_1, \theta_2 \to -\pi} \sqrt{r_1 r_2} e^{\frac{1}{2} i(\theta_1+\theta_2)} = \sqrt{r_1 r_2} e^{-\pi i} = -\sqrt{r_1 r_2}$$

These limits are the same so the limit is well-defined. Therefore, $f(z)$ is a continuous, single-valued function with a branch cut on the real numbers from -1 to 1.

4 Ch. 14, §1, p. 667, 7

$$\begin{aligned}
\cosh z &= \frac{e^z + e^{-z}}{2} = \frac{e^{x+iy} + e^{-(x+iy)}}{2} && (4.1) \\
&= \frac{e^x(\cos(y) + i\sin(y)) + e^{-x}(\cos(-y) + i\sin(-y))}{2} && (4.2) \\
&= \frac{e^x(\cos(y) + i\sin(y)) + e^{-x}(\cos(y) - i\sin(y))}{2} && (4.3) \\
&= \cos(y)(\frac{e^x + e^{-x}}{2}) + i\sin(y)(\frac{e^x - e^{-x}}{2}) && (4.4) \\
&= \cos(y)\cosh(x) + i\sin(y)\sinh(x) && (4.5) \\
\Rightarrow u &= \cosh(x)\cos(y) && (4.6) \\
v &= \sinh(x)\sin(y) && (4.7)
\end{aligned}$$

5 Ch. 14, §1, p. 667, 15

$$\begin{aligned}
\overline{e^z} &= e^{\overline{z}} = e^x e^{-iy} = e^x(\cos y - i\sin y) && (5.1) \\
u &= e^x \cos y && (5.2) \\
v &= -e^x \sin y && (5.3)
\end{aligned}$$

6 Ch. 14, §1, p. 667, 18

$$\begin{aligned}
\sqrt{z} &= \sqrt{r}e^{i\theta/2} = \sqrt{r}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) && (6.1)\\
u &= \sqrt{r}\cos\frac{\theta}{2} && (6.2)\\
v &= \sqrt{r}\sin\frac{\theta}{2} && (6.3)
\end{aligned}$$

7 Ch. 14, §2, p. 672, 23

$$\begin{aligned}
f &= \frac{x - iy}{x^2 + y^2} && (7.1)\\
u &= \frac{x}{x^2 + y^2} && (7.2)\\
v &= \frac{-y}{x^2 + y^2} && (7.3)\\
\partial_x u &= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} && (7.4)\\
\partial_y v &= \frac{-x^2 - y^2 + 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \partial_x u && (7.5)\\
\partial_x v &= \frac{2yx}{(x^2 + y^2)^2} && (7.6)\\
\partial_y u &= \frac{-2xy}{(x^2 + y^2)^2} = -\partial_x v && (7.7)
\end{aligned}$$

Thus, the Cauchy-Riemann conditions are satisfied so $f(z)$ is analytic.

8 Ch. 14, §2, p. 673, 45

$$\begin{aligned}
\mathbf{F}(z) &= v\hat{\mathbf{i}} + u\hat{\mathbf{j}} && (8.1)\\
\vec{\nabla} \cdot \mathbf{F} &= \partial_x v + \partial_y u = 0 && (8.2)\\
\Rightarrow \partial_x v &= -\partial_y u && (8.3)\\
\vec{\nabla} \times \mathbf{F} &= (\partial_x u - \partial_y v)\hat{\mathbf{k}} = 0 && (8.4)\\
\Rightarrow \partial_x u &= \partial_y v && (8.5)
\end{aligned}$$

9 Ch. 14, §2, p. 674, 58

$$\nabla^2 u = (\partial_x^2 + \partial_y^2)\cosh y \cos x = -\cosh y \cos x + \cosh y \cos x = 0$$

From the third problem we have that $\cosh y \cos x = \Re \cosh(y - ix)$. Thus, if $f(z) = \cosh(-iz) = \cosh(y - ix)$ then $u = \Re[f(z)]$ and $v = \Im[f(z)] = -\sinh y \sin x$. Also,

$$\nabla^2 v = \sinh y \sin x - \sinh y \sin x = 0$$

10 Ch. 14, §2, p. 674, 61

$$u(x, y) = \frac{x}{x^2 + y^2}$$

$$\begin{aligned}(\partial_x^2 + \partial_y^2)u = (\partial_x + i\partial_y)(\partial_x - i\partial_y)u &= (\partial_x + i\partial_y)\frac{(x^2 + y^2) - x(2x - 2iy)}{(x^2 + y^2)^2} \\ = (\partial_x + i\partial_y)\frac{(ix + y)^2}{(y + ix)^2(y - ix)^2} &= (\partial_x + i\partial_y)\frac{-1}{(x + iy)^2} = -2(x + iy)^{-3}(1 - 1) = 0\end{aligned}$$

Therefore, u is harmonic. Now find v using Cauchy-Riemann conditions:

$$\partial_x v = -\partial_y u = \frac{2xy}{(x^2 + y^2)^2}$$

so

$$v = \frac{-y}{x^2 + y^2} + g(y)$$

where g is unknown. To find g, use the other condition:

$$\begin{gathered}\partial_x u = \partial_y v \\ \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + g'(y)\end{gathered}$$

so $g(y)$ is an arbitrary constant which we can set to zero. Now show that v is harmonic:

$$v(x, y) = \frac{-y}{x^2 + y^2}$$

$$\begin{aligned}(\partial_x^2 + \partial_y^2)v = (\partial_x + i\partial_y)(\partial_x - i\partial_y)v &= (\partial_x + i\partial_y)\frac{(x^2 + y^2)i + y(2x - 2iy)}{(x^2 + y^2)^2} = (\partial_x + i\partial_y)\frac{i(x - iy)^2}{(x + iy)^2(x - iy)^2} \\ = (\partial_x + i\partial_y)\frac{i}{(x + iy)^2} &= -2i(x + iy)^{-3}(1 - 1) = 0\end{aligned}$$

Note: Prof. Marolf pointed out that it's actually easier to find f and v by inspection. Consider, for example, the following reasoning:

$$Re(f(z)) = \frac{x}{x^2 + y^2} = \frac{x}{z\overline{z}}$$

so

$$f(z) = \frac{x + ih(x,y)}{z\overline{z}}$$

for some function h which we must determine. We know analytic functions are only functions of z, not $\overline{z}$, so if we choose h such that the numerator cancels the $\overline{z}$ in the denominator, then f will be analytic. Therefore, let $h(x,y) = -y$. Then,

$$f(z) = \frac{x - iy}{z\overline{z}} = \frac{1}{z}$$

and

$$v(x,y) = Im\left(\frac{1}{z}\right) = Im\left(\frac{1}{x+iy}\right) = Im\left(\frac{x-iy}{x^2+y^2}\right) = \frac{-y}{x^2+y^2}$$

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Now consider the function $u(x,y) = x^2 + y^2$. Is it the real part of an analytic function? $(\partial_x^2 + \partial_y^2)u = 2 + 2 \neq 0$ so u is not the real part of an analytic function. This is, of course, just the contrapositive of the statement "If u is the real part of an analytic function then u is harmonic."

12 Ch. 14, §3, p. 677, 10

We want to evaluate

$$\int_2^{2+i\infty} ze^{iz}dz \tag{12.1}$$

Along this contour, $z = 2 + iy$, so $dz = idy$.

$$\begin{aligned}
\int_2^{2+i\infty} ze^{iz}dz &= \int_0^\infty (2+iy)e^{i(2+iy)}idy && (12.2)\\
&= \int_0^\infty \left(2ie^{2i}e^{-y} - e^{2i}ye^{-y}\right)dy && (12.3)\\
&= \left[-2ie^{2i}e^{-y}\right]_0^\infty + \left[e^{2i}ye^{-y}\right]_0^\infty - e^{2i}\int_0^\infty e^{-y}dy && (12.4)\\
&= 2ie^{2i} + \left[e^{2i}e^{-y}\right]_0^\infty && (12.5)\\
&= (2i-1)e^{2i} && (12.6)
\end{aligned}$$

13 Ch. 14, §3, p. 677, 14

$\oint_C z^{n-m-1}dz = 0$ by Cauchy's Theorem when $n > m$ since z^α is analytic for $\alpha \in \mathbb{N}$. Let $z = e^{ix}$ so that $dz = ie^{ix}dx = izdx$ and $z^{n-m-1}dz = z^n z^{-m} z^{-1}dz = ie^{inx}e^{-imx}dx$. If C is the circle $|z| = 1$ then the range of our integral is $0 < x < 2\pi$ so we have that $\int_0^{2\pi} e^{inx}e^{-imx}dx = \frac{1}{i}\oint_C z^{n-m-1}dz = 0$. Note: The case $n < m$ can be treated similarly by taking the conjugate of $\int_0^{2\pi} e^{inx}e^{-imx}dx$.

14 Ch. 14, §3, p. 677, 15

We know that $f(z)$ is analytic in an open set containing $|z| \leq 1$, so we know $\oint_{|z|=1} f(z)dz = 0$ by Cauchy's theorem. We can choose to parametrize the curve $|z| = 1$ by $z = e^{i\theta}$, $\theta \in [0, 2\pi)$. With this parametrization, $dz = ie^{i\theta}d\theta$. Thus,

$$\oint_{|z|=1} f(z)dz = \int_0^{2\pi} f(e^{i\theta})ie^{i\theta}d\theta = 0 \tag{14.1}$$

Therefore, dividing by i, which does not effect analyticity, we find the integral we are looking for.