

# Physics 101 Homework 3 Solutions

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January 23, 2013

## 1 Ch. 14, §3, p. 677, 17

We want to evaluate the contour integral

$$\int_C \frac{\sin z}{2z - \pi} dz \quad (1.1)$$

about two different curves,  $|z| = 1$  and  $|z| = 2$ . Now,  $\oint_C \frac{\sin z}{2z - \pi} dz = \frac{1}{2} \oint_C \frac{\sin z}{z - \frac{\pi}{2}} dz$  so there is a single (simple) pole at  $z = \frac{\pi}{2} \approx 1.57$ . The contour  $|z| = 1$  does not contain this pole - the function is analytic everywhere *within* the contour so the integral for this case is 0 by Cauchy's Theorem.

However, our pole is within the contour  $|z| = 2$  so by Cauchy's Integral Formula, for this contour we have:

$$\oint_{|z|=2} \frac{\sin z}{2z - \pi} dz = \frac{1}{2} \oint_{|z|=2} \frac{\sin z}{z - \frac{\pi}{2}} dz = \frac{1}{2} 2\pi i \sin\left(\frac{\pi}{2}\right) = \pi i$$

## 2 Ch. 14, §3, p. 678, 22

We will use the following special case ( $n = 2$ ) of the result from problem 21:

$$f^{(2)}(z) \equiv \frac{\partial^2 f}{\partial z^2} = \frac{1}{\pi i} \oint_C \frac{f(w)dw}{(w - z)^3}$$

Now, we must simply manipulate the integral we are given into the above form. Also remember that the pole must still lie within the contour, otherwise the integral will be zero by Cauchy's Theorem.

$$\oint_{|z|=3} \frac{\sin 2z dz}{(6z - \pi)^3} = \oint \frac{\sin 2z dz}{6^3(z - \pi/6)^3} = \frac{\pi i}{6^3} \left. \frac{\partial^2 \sin(2z)}{\partial z^2} \right|_{z=\pi/6} = \frac{-4\pi i \frac{\sqrt{3}}{2}}{6^3} = \frac{-\sqrt{3}\pi i}{108} \quad (2.1)$$

## 3 Ch. 14, §4, p. 681, 5

We want all Laurent series about the origin for

$$\frac{z - 1}{z^3(z - 2)} \quad (3.1)$$

This function has a pole of order 3 at the origin and a simple pole at  $z = 2$ . First, we consider the Laurent series in the annulus  $0 < |z| < 2$ .

$$\frac{z-1}{z^3(z-2)} = \frac{1-z}{2z^3} \frac{1}{1-z/2} = \frac{1-z}{2z^3} \sum_{n=0}^{\infty} \frac{z^n}{2^n} = (1-z) \sum_{n=0}^{\infty} \frac{z^{n-3}}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{z^{n-3}}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{z^{n-3}}{2^n} = \frac{1}{2z^3} - \sum_{n=1}^{\infty} \frac{z^{n-3}}{2^{n+1}} \quad (3.2)$$

Thus, the residue is  $-1/8$ .

Now we can consider the Laurent series in the annulus  $2 < |z|$ . Let us make the change of variables  $w = \frac{1}{z}$  and consider the series in the disk  $|w| < 1/2$ , then at the end of the calculation change back to  $z$ :

$$\frac{z-1}{z^3(z-2)} = \frac{w^3(1-w)}{1-2w} = w^3(1-w) \sum_{n=0}^{\infty} (2w)^n = w^3 + \sum_{n=0}^{\infty} w^4(2w)^n = \frac{1}{z^3} + \sum_{n=0}^{\infty} \frac{2^n}{z^{4+n}} \quad (3.3)$$

## 4 Ch. 14, §4, p. 682, 10

(a)

$$\frac{e^z - 1}{z^2 + 4} = \frac{e^z - 1}{(z + 2i)(z - 2i)} \quad (4.1)$$

As we can see, this has a simple pole at  $z = 2i$  (and also one at  $z = -2i$ ).

(b)

$$\tan^2 z = \frac{\sin^2 z}{\cos^2 z} = \frac{\sin^2 z}{(z - \pi/2)^2 + \dots} \quad (4.2)$$

Since  $\sin^2 z$  is regular at  $z = \pi/2$ , we simply need to expand the denominator, which gives  $(z - \pi/2)^2 + \dots$ , so there is a pole of order 2.

(c)

$$\frac{1 - \cos z}{z^4} = \frac{1 - 1 + z^2/2 - z^4/4! + \dots}{z^4} = \frac{1}{2z^2} - \frac{1}{4!} + \dots \quad (4.3)$$

Thus, there is a pole of order 2 at  $z = 0$  (and the residue is zero).

(d)

Let  $w = \frac{1}{z-\pi}$ . Then since  $\cos(\pi w) = 1 - \frac{\pi^2 w^2}{2} + \frac{\pi^4 w^4}{24} \dots$  we have that:

$$\cos\left(\frac{\pi}{z-\pi}\right) = 1 - \frac{\pi^2}{2(z-\pi)^2} + \frac{\pi^4}{24(z-\pi)^4} \dots \quad (4.4)$$

So the Laurent Series has infinite order and we therefore have an essential singularity at  $z = \pi$ .

## 5 Ch. 14, §5, p. 683, 1

First consider the case  $n = 1$ , for which it is clear that since the residue is 1, the integral evaluates to  $2\pi i$ . Now consider the case  $n \neq 1$ . If we choose the parametrization  $z = z_0 + \rho e^{i\theta}$ ,  $dz = i\rho e^{i\theta} d\theta$ .

$$\oint_C \frac{dz}{(z - z_0)^n} = \int_0^{2\pi} i\rho^{1-n} e^{i(1-n)\theta} d\theta = \frac{i\rho^{1-n}}{i(1-n)} \left[ e^{i\theta(1-n)} \right]_0^{2\pi} = \frac{\rho^{1-n}}{n-1} (1 - e^{2\pi i(1-n)}) = 0 \quad (5.1)$$

**6 Ch. 14, §6, p. 686, 3**

$$\frac{\sin z}{z^4} = z^{-4} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-3}}{(2n+1)!} \quad (6.1)$$

Thus, the residue is  $-1/3! = -1/6$ .

**7 Ch. 14, §6, p. 686, 7**

$$\frac{\sin \pi z}{4z^2 - 1} = \frac{1}{z - 1/2} \frac{\cos \pi(z - 1/2)}{4z + 2} = \frac{1}{z - 1/2} \frac{1/4}{1 + (z - 1/2)} \cos \pi(z - 1/2) \quad (7.1)$$

$$= \frac{1}{4} (z - 1/2)^{-1} \left( \sum_{n=0}^{\infty} (-1)^n (z - 1/2)^n \right) \left( \sum_{m=0}^{\infty} (-1)^m \frac{(\pi(z - 1/2))^{2m}}{(2m)!} \right) \quad (7.2)$$

$$= \frac{1}{4(z - 1/2)} - \frac{1}{4} + \left( \frac{1}{4} - \frac{\pi^2}{8} \right) (z - 1/2) + \dots \quad (7.3)$$

Thus, we see that the residue is  $1/4$ .

**8 Ch. 14, §11, p. 718, 2**

We want to show that  $\nabla^2 u(x, y) = 0$ , namely  $u_{xx} + u_{yy} = 0$ .

$$u(x, y) = \tan^{-1} \left( \frac{y}{x+1} \right) = \frac{1}{2i} \left( \log \left( 1 + \frac{iy}{x+1} \right) - \log \left( 1 - \frac{iy}{x+1} \right) \right)$$

$$u_x = \frac{-y}{(x+1)^2 + y^2}$$

$$u_{xx} = \frac{2y(x+1)}{((x+1)^2 + y^2)^2}$$

$$u_y = \frac{x+1}{(x+1)^2 + y^2}$$

$$u_{yy} = \frac{-2y(x+1)}{((x+1)^2 + y^2)^2}$$

$$\nabla^2 u(x, y) = 0$$

Now we want to find a harmonic conjugate for  $u$  using the Cauchy-Riemann equations,  $u_x = v_y, u_y = -v_x$ .

$$\begin{aligned} v_y &= \frac{-y}{(x+1)^2 + y^2} \\ v(x, y) &= -\frac{1}{2} \log((x+1)^2 + y^2) + g(x) \\ v_x &= \frac{-x-1}{(x+1)^2 + y^2} \\ v(x, y) &= -\frac{1}{2} \log((x+1)^2 + y^2) + h(y) \\ v(x, y) &= -\frac{1}{2} \log((x+1)^2 + y^2) + c \\ f(z) &= \arctan\left(\frac{y}{x+1}\right) - \frac{i}{2} \log((x+1)^2 + y^2) + c \\ &= -i \log(z+1) + c \end{aligned}$$

## 9 Ch. 14, §11, p. 718, 3

Assume that  $f(z)$  is analytic for all  $z$  and  $|f(z)| \leq M$  for some constant  $M$ . By the result of problem 3.21,  $f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)dw}{(w-z)^2}$ . Let the contour  $C$  be the circle of radius  $R$  centered on  $z$ . Then we have:

$$|f'(z)| = \left| \frac{1}{2\pi i} \oint_C \frac{f(w)dw}{(w-z)^2} \right| \leq \frac{1}{2\pi} \oint_C \left| \frac{f(w)dw}{(w-z)^2} \right| = \frac{1}{2\pi} \oint_C \frac{|f(w)||dw|}{|w-z|^2} \leq \frac{1}{2\pi} \oint_C \frac{M|dw|}{R^2} = \frac{M}{2\pi R^2} \oint_C |dz|$$

But since  $\oint_C |dz|$  just measures arc length around  $C$ , the integral is the circumference of the circle of radius  $R$ , so we arrive at the inequality  $|f'(z)| \leq \frac{M}{R}$ . Since  $R$  was arbitrary, it can be as large as we like so  $f'(z) = 0$ . Therefore,  $f$  must be constant.

## 10 Ch. 14, §11, p. 720, 36

Any harmonic function is the real part of a holomorphic function. Therefore, let the harmonic function  $u$  be the real part of the holomorphic function  $f(z)$ . First show that  $f(a)$  is the average value of the contour integral on a circle  $z = a + re^{i\theta}$  about the point  $a$ . We'll start with Cauchy's Integral Formula and do the usual change of variables:

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \oint \frac{f(z)dz}{z-a} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{i\theta} f(a + re^{i\theta})d\theta}{re^{i\theta}} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})d\theta \end{aligned}$$

Now take the real part of both sides:

$$u(a) = \frac{1}{2\pi r} \int_0^{2\pi} u(r, \theta) r d\theta$$

This is the usual formula for the average of a function over a (one dimensional) region, so  $u(a)$  is the average of  $u$  on any circle centered at  $a$  and lying in the region where  $f(z)$  is holomorphic.

## 11 Ch. 14, §11, p. 720, 37

Most of the paragraph describing problem 37 is just introduction. What we are asked to do is stated in the last few lines: We are given that a real function  $u(x, y)$  is harmonic. We are also given that all points in some small disk  $D$  (say, of radius  $R$ ) about  $a$  we have  $u(a) \geq u(x, y)$  for  $(x, y) \in D$ . We are also told that  $u(x, y)$  is not constant on  $D$  and asked to show a contradiction.

This is easy using #36 above:

$$u(a) = \frac{1}{2\pi r} \int_0^{2\pi} u(r, \theta) r d\theta, \tag{11.1}$$

where the integral is over any circle centered on  $a$  (i.e., concentric with the disk  $D$  above).

Suppose there is some point  $b \in D$  such that  $u(b) \neq u(a)$ . Then  $u(b) < u(a)$  by the above. In fact, since  $u$  is harmonic, it is continuous and  $u(x, y) < u(a)$  for all  $(x, y)$  close enough to  $b$ .

Let us then apply (11.1) to a circle that passes through  $b$ . Then  $u(x, y) < u(a)$  for some range  $\Delta\theta > 0$  of the integration variable  $\theta$ . For other values of  $\theta$ , we were already given  $u(x, y) \leq u(a)$ . So (11.1) gives

$$u(a) < \left( \frac{1}{2\pi r} \right) (r\Delta\theta u(a) + (2\pi - \Delta\theta)ru(a)) = u(a).$$

I.e.,  $u(a) < u(a)$ , which is the desired contradiction.

The proof that  $u$  also takes its minimum value on the boundary is identical. The short version is to note that if  $u$  is harmonic, then so is  $-u$ . And the minimum of  $u$  occurs at the maximum of  $-u$ . But we proved above that this value occurs on the boundary of any region.