# Physics 101 Homework 4 Solutions

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## 1 Ch. 14, §6, p. 686, 16

We want the residues of

$$\frac{z-2}{z(1-z)} = \frac{2-z}{z(z-1)} \tag{1.1}$$

at $z = 0$ and $z = 1$. Since these are both simple poles, we can simply multiply by $(z - z_0)$ and evaluate at $z_0$. For $z_0 = 0$, this gives

$$\left.\frac{z-2}{1-z}\right|_{z=0} = -2 \tag{1.2}$$

For $z_0 = 1$, this gives

$$\left.\frac{2-z}{z}\right|_{z=1} = 1 \tag{1.3}$$

## 2 Ch. 14, §6, p. 687, 26

$$\frac{e^{2\pi iz}}{1-z^3} = \frac{e^{2\pi iz}}{(1-z)(z-e^{2\pi i/3})(z-e^{4\pi i/3})} \tag{2.1}$$

Again, we have a simple pole, so to find the residue, we simply multiply by $(z - e^{2\pi i/3})$ and evaluate at $z = e^{2\pi i/3}$, which gives

$$\frac{e^{2\pi i e^{2\pi i/3}}}{(1-e^{2\pi i/3})(e^{2\pi i/3}-e^{4\pi i/3})} = \frac{e^{-\sqrt{3}\pi+2\pi i/3}}{3} \tag{2.2}$$

## 3 Ch. 14, §6, p. 687, 29

We want to find the residue of $\frac{e^{2z}}{4\cosh z-5}$ at $z = \log 2$. Since the numerator is regular at $z = \log 2$, we will use the formula $res_{z_0}\frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$.

$$\left.\frac{g(z)}{h'(z)}\right|_{\log 2} = \frac{e^{2\log 2}}{4\sinh\log 2} = \frac{e^{\log 2^2}}{2(e^{\log 2}-e^{-\log 2})} = \frac{4}{2(2-1/2)} = \frac{4}{3} \tag{3.1}$$

## 4 Ch. 14, §6, p. 687, 26'

We want to find the contour integral of $\frac{e^{2\pi iz}}{1-z^3}$ on the contour of radius 3/2 about the origin. This circle encloses all three poles once, so we pick up the residue at each pole once.

$$\frac{e^{2\pi iz}}{1-z^3} = \frac{-e^{2\pi iz}}{(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})} \tag{4.1}$$

$$res_1 = \frac{-e^{2\pi i}}{(1-e^{2\pi i/3})(1-e^{4\pi i/3})} = -1/3 \tag{4.2}$$

$$res_{e^{2\pi i/3}} = \frac{-e^{2\pi i e^{2\pi i/3}}}{(e^{2\pi i/3}-1)(e^{2\pi i/3}-e^{4\pi i/3})} \tag{4.3}$$

$$res_{e^{4\pi i/3}} = \frac{-e^{2\pi i e^{4\pi i/3}}}{(e^{4\pi i/3}-1)(e^{4\pi i/3}-e^{2\pi i/3})} \tag{4.4}$$

$$\oint_{|z|=3/2} \frac{e^{2\pi iz}dz}{1-z^3} = 2\pi i\left(-\frac{1}{3} + \frac{-e^{2\pi i e^{2\pi i/3}}}{(e^{2\pi i/3}-1)(e^{2\pi i/3}-e^{4\pi i/3})} + \frac{-e^{2\pi i e^{4\pi i/3}}}{(e^{4\pi i/3}-1)(e^{4\pi i/3}-e^{2\pi i/3})}\right) \tag{4.5}$$

$$= -\frac{2\pi i}{3}\left(1+\cosh\sqrt{3}\pi + \sqrt{3}i\sinh\sqrt{3}\pi\right) \tag{4.6}$$

## 5 Ch. 14, §6, p. 687, 35'

$$\oint_{|z|=3/2} \frac{z}{(z^2+1)^2}dz = \oint_{|z|=3/2} \frac{z}{(z-i)^2(z+i)^2}dz \tag{5.1}$$

$$\tag{5.2}$$

Thus, the poles at $z = i$ and $z = -i$ are both second-order, so to compute the residues we use the technique described on page 685. For a second order pole at $z = z_0$ the formula gives that $Res[z_0] = \left.\frac{\partial[(z-z_0)^2 f]}{\partial z}\right|_{z=z_0}$. This yields:

$$res_i = \partial_z \left.\frac{z}{(z+i)^2}\right|_{z=i} = 0 \tag{5.3}$$

$$res_{-i} = \partial_z \left.\frac{z}{(z-i)^2}\right|_{z=-i} = 0 \tag{5.4}$$

$$\oint_{|z|=3/2} \frac{z}{(z^2+1)^2}dz = 2\pi i(0+0) = 0 \tag{5.5}$$

## 6 Extra Problem

To evaluate $\oint_{|z|=2} \frac{e^z}{\cosh z}dz$ we must find residues for poles with $|z| < 2$. Since $e^z$ is entire and non-zero, the poles occur when $\cosh z = 0 \Rightarrow \frac{e^z+e^{-z}}{2} = 0 \Rightarrow e^{2z} = -1 \Rightarrow 2z = n\pi i$ for $n$ an odd integer. Thus, the only poles in our contour are $z = \frac{\pi i}{2}$ and $z = \frac{-\pi i}{2}$. Consideration of the Taylor series for $e^z$ suggests that these are simple poles, so we will try to find the residues using equation

14.6.2:

$$Res(z = \frac{\pi i}{2}) = \left.\frac{e^z}{\partial_z[\cosh(z)]}\right|_{z=\frac{\pi i}{2}} = \frac{e^{\frac{\pi i}{2}}}{\sinh(\frac{\pi i}{2})} = \frac{i}{i} = 1$$

$$Res(z = \frac{-\pi i}{2}) = \left.\frac{e^z}{\partial_z[\cosh(z)]}\right|_{z=\frac{-\pi i}{2}} = \frac{e^{-\frac{\pi i}{2}}}{\sinh(-\frac{\pi i}{2})} = \frac{-i}{-i} = 1$$

Since these are finite, our guess that the poles were simple was correct. Now, the residue theorem yields:

$$\oint_{|z|=2} \frac{e^z}{\cosh z} dz = 2\pi i \sum[\text{Residues inside } C] = 2\pi i(1+1) = 4\pi i$$

# 7 Ch. 14, §7, p. 699, 8

Since $\frac{\sin^2\theta}{13-12\cos\theta}$ is even and periodic with period $2\pi$, it is clear that $\int_0^\pi \frac{\sin^2\theta d\theta}{13-12\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta d\theta}{13-12\cos\theta}$. Now, we make the standard change of variables $z = e^{i\theta}$ to get the following contour integral:

$$\frac{1}{2}\oint_{|z|=1} \frac{(1/2i)^2(z-1/z)^2}{13-6(z+1/z)} dz = \frac{1}{8i}\oint \frac{(z^2-1)^2 dz}{z^2(2z-3)(3z-2)}$$

We see that there are simple poles at $z = \frac{3}{2}$ and $z = \frac{2}{3}$, as well as a second order pole at $z = 0$. To apply the residue theorem we must calculate the residues at all poles within our contour; namely, residues at poles with $|z| < 1$:

$$res_{2/3} = \left.\frac{(z^2-1)^2}{3(2z-3)z^2}\right|_{z=2/3} = -5/36$$

$$res_0 = \partial_z \left.\frac{(z^2-1)^2}{6z^2-13z+6}\right|_{z=0} = 13/36$$

Thus, the residue theorem yields:

$$\int_0^\pi \frac{\sin^2\theta d\theta}{13-12\cos\theta} = \frac{2\pi i}{8i}(13/36 - 5/36) = \pi/18$$

# 8 Ch. 14, §7, p. 699, 11

First, note that $\int_0^\infty \frac{dx}{(4x^2+1)^3} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(4x^2+1)^3}$ since the integrand is even. Also, since the integrand falls off more rapidly than $\frac{1}{x^2}$, we may integrate a semi-circular contour about the upper half-plane and the contribution along the arc vanishes in the limit that the radius goes to infinity. Thus:

$$\int_0^\infty \frac{dx}{(4x^2+1)^3} = \frac{1}{2}\oint_\Gamma \frac{dz}{(4z^2+1)^3} = \frac{1}{128}\oint_\Gamma \frac{dz}{(z-i/2)^3(z+i/2)^3}$$

The only residue in our contour is at $z = i/2$:

$$res_{i/2} = \frac{1}{2}\partial_z^2 \left.\frac{1}{(z+i/2)^3}\right|_{z=i/2} = \left.\frac{6}{(z+i/2)^5}\right|_{z=i/2} = -6i$$

So the residue theorem gives:

$$\int_0^\infty \frac{dx}{(4x^2+1)^3} = \frac{1}{128}\oint_\Gamma \frac{dz}{(z-i/2)^3(z+i/2)^3} = \frac{1}{128}2\pi i(-6i) = \frac{3\pi}{32}$$

# 9 Ch. 14, §7, p. 699, 20

As in the previous problem, we note that our function is even so we may extend the integral over the whole real line and add a factor of $\frac{1}{2}$. Also, the integral is purely real so using that $\cos z = \mathrm{Re}(e^{iz})$ we may write:

$$\int_0^\infty \frac{\cos x dx}{(1+9x^2)^2} = \mathrm{Re}\left(\frac{1}{2}\int_{-\infty}^\infty \frac{e^{ix}dx}{(1+9x^2)^2}\right) = \mathrm{Re}\left(\frac{1}{2}\oint_\Gamma \frac{e^{iz}dz}{81(z-i/3)^2(z+i/3)^2}\right)$$

However, we must be careful. We want to close the contour with an arc that doesn't contribute to the integral, and to guarantee this, we seek to pick a contour in which the integrand dies as $z$ goes to infinity. If we choose the lower half plane, then along the negative imaginary axis $e^{iz} = e^{i(-iy)} = e^y$ which diverges for $y \to \infty$. Thus, we must be sure to close our contour in the upper half-plane, where the integrand dies out rapidly. In this region we have a second order pole at $z = i/3$. The residue is:

$$res_{i/3} = \partial_z \left.\frac{e^{iz}}{81(z+i/3)^2}\right|_{z=i/3} = \frac{-i}{9e^{1/3}}$$

Therefore, the residue theorem gives that:

$$\int_0^\infty \frac{\cos x dx}{(1+9x^2)^2} = \mathrm{Re}\left(\frac{1}{2}\oint_\Gamma \frac{e^{iz}dz}{81(z-i/3)^2(z+i/3)^2}\right) = \mathrm{Re}\left(\frac{1}{2}2\pi i\frac{-i}{9e^{1/3}}\right) = \frac{\pi}{9e^{\frac{1}{3}}}$$

# 10 Ch. 14, §11, p. 719, 21

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_{|z|=1} \frac{-idz}{z(a+b(\frac{z-z^{-1}}{2i}))} = \oint_{|z|=1} \frac{2dz}{bz^2+2iaz-b}$$

There are poles at $z = \frac{-2ia\pm\sqrt{-4a^2+4b^2}}{2b} = \frac{-ia\pm\sqrt{b^2-a^2}}{b}$ so the only pole inside the contour is at $\frac{-ia+\sqrt{b^2-a^2}}{b}$. The residue is:

$$\left.\frac{2}{b\left(z-\frac{-ia-\sqrt{b^2-a^2}}{b}\right)}\right|_{\frac{-ia+\sqrt{b^2-a^2}}{b}} = \frac{1}{\sqrt{b^2-a^2}} = \frac{-i}{\sqrt{a^2-b^2}}$$

Thus, the residue theorem gives that $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$. Now, by the periodicity of $\sin z$, we may shift $\theta$ by $\pi/2$ without changing the result. But $\sin(\theta+\pi/2) = \cos\theta$ so $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ also.

## 11 Ch. 14, §11, p. 720, 24

$$\int_0^\infty \frac{\cos mx dx}{x^2+a^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos mx dx}{x^2+a^2} = \text{Re}\left(\frac{1}{2}\int_{-\infty}^\infty \frac{e^{imx}dx}{x^2+a^2}\right) = \text{Re}\left(\frac{1}{2}\oint_\Gamma \frac{e^{imx}dx}{x^2+a^2}\right)$$

Here, we have chosen $\Gamma$ to be the upper-half plane. Since the function decays rapidly along the positive imaginary axis but diverges along the negative imaginary axis, closing the contour with a semi-circular piece is only guaranteed to give the same integral for the upper-half plane. There are poles at $z = \pm ia$ so the only pole contained in $\Gamma$ is at $z = ia$. Since it is a simple pole, we see immediately that the residue is $\frac{e^{im(ia)}}{ia+ia} = \frac{e^{-ma}}{2ia}$. Thus, the residue theorem gives:

$$\frac{1}{2}\oint_\Gamma \frac{e^{imx}dx}{x^2+a^2} = \frac{1}{2}2\pi i\frac{e^{-ma}}{2ia} = \frac{\pi e^{-ma}}{2a}$$

Using our original equality shows:

$$\int_0^\infty \frac{\cos mx dx}{x^2+a^2} = \text{Re}\left(\frac{1}{2}\oint_\Gamma \frac{e^{imx}dx}{x^2+a^2}\right) = \text{Re}\left(\frac{\pi e^{-ma}}{2a}\right) = \frac{\pi e^{-ma}}{2a}$$