# Physics 101 Homework 5 Solutions

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## 1 Ch. 14, §7, p. 700, 25

$$PV\int_0^\infty \frac{x\sin x dx}{9x^2-\pi^2} = PV\frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x dx}{9x^2-\pi^2} = \frac{1}{2}\Im\left[PV\oint_{\texttt{UHP}} \frac{ze^{iz}}{9(z+\pi/3)(z-\pi/3)}\right] \quad (1.1)$$

$$res_{\pi/3} = \frac{\pi e^{\pi i/3}/3}{9(2\pi/3)} = \frac{e^{\pi i/3}}{18} \quad (1.2)$$

$$res_{-\pi/3} = \frac{-\pi e^{-\pi i/3}/3}{9(-2\pi/3)} = \frac{e^{-\pi i/3}}{18} \quad (1.3)$$

$$\Rightarrow \int_0^\infty \frac{x\sin x dx}{9x^2-\pi^2} = \frac{1}{2}\Im\left[\pi i\frac{(e^{\pi i/3}+e^{-\pi i/3})}{18}\right] = \pi/36 \quad (1.4)$$

## 2 Ch. 14, §7, p. 700, 26

$$\int_{-\infty}^\infty \frac{xdx}{(x-1)^4-1} = \oint_{\texttt{UHP}} \frac{zdz}{z(z-2)(z-i-1)(z+i-1)} = \oint_{\texttt{UHP}} \frac{dz}{(z-2)(z-i-1)(z+i-1)} \quad (2.1)$$

$$res_2 = \frac{1}{(1-i)(1+i)} = 1/2 \quad (2.2)$$

$$res_{1+i} = \frac{1}{(i-1)2i} = (i-1)/4 \quad (2.3)$$

$$\Rightarrow \int_{-\infty}^\infty \frac{xdx}{(x-1)^4-1} = 2\pi i(1/4+(i-1)/4) = -\pi/2 \quad (2.4)$$

Note that we took $\frac{1}{2}$ times the residue on the contour, according to the principal value convention.

## 3 Ch. 14, §7, p.700, 34

$$I = \int_0^\infty \frac{\sqrt{x}dx}{(1+x)^2} \quad (3.1)$$

$$J = \oint_\Gamma \frac{\sqrt{z}dz}{(1+z)^2} \quad (3.2)$$

The function $\frac{\sqrt{z}}{(1+z)^2}$ has second order branch points at 0 and $\infty$. We will take the branch cut to lie along the positive real axis and consider a contour $\Gamma$ which circles the complex plane at a radius $R$ along $\gamma_R$ and dodges the branch cut, going in just below and out just above the positive real axis along $\gamma_{in}$ and $\gamma_{out}$, respectively, and circles the branch cut at a radius $\epsilon$ along $\gamma_\epsilon$ (see figure 1). The function also has a pole of order 2 at $z=-1$, which $\Gamma$ encloses once. We can choose to parametrize $\gamma_{out}$ by $z=t, 0<t<R$ and $\gamma_{in}$ by $z=se^{2\pi i}$, where $s$ ranges from $R$ to 0.

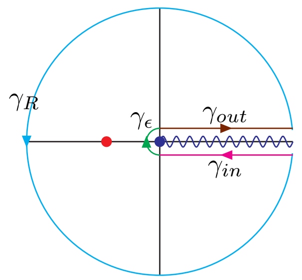



Figure 1: The contour $\Gamma$ is composed of $\gamma_R, \gamma_{in}, \gamma_{out}$ and $\gamma_\epsilon$ in the complex $z$ plane. The pole at $z=-1$ is shown in red while the branch cut and branch point at 0 are shown in blue.

$$\left|\int_{\gamma_R} \frac{\sqrt{z}dz}{(1+z)^2}\right| \leq 2\pi R\frac{\sqrt{R}}{R^2} \to 0, R\to\infty \tag{3.3}$$

$$\left|\int_{\gamma_R} \frac{\sqrt{z}dz}{(1+z)^2}\right| \leq 2\pi\epsilon\frac{\sqrt{\epsilon}}{(1+\epsilon)^2} \leq 2\pi\epsilon^{3/2} \to 0, \epsilon\to 0 \tag{3.4}$$

$$J = \int_0^\infty \frac{\sqrt{t}dt}{(1+t)^2} - \int_0^\infty \frac{\sqrt{s}e^{i\pi}ds}{(1+s)^2} = (1-e^{i\pi})I \tag{3.5}$$

$$J = 2\pi i res_{-1}\left(\frac{\sqrt{z}}{(1+z)^2}\right) \tag{3.6}$$

$$res_{-1}\left(\frac{\sqrt{z}}{(1+z)^2}\right) = \partial_z\sqrt{z}\big|_{z=-1} = -i/2 \tag{3.7}$$

$$\Rightarrow I = \int_0^\infty \frac{\sqrt{x}dx}{(1+x)^2} = \frac{2\pi i(-i)}{2(1-e^{i\pi})} = \pi/2 \tag{3.8}$$

# 4 Ch. 14, §7, p. 700, 36

$$I = \int_0^\infty \frac{\log x dx}{x^{3/4}(1+x)} \tag{4.1}$$

$$J = \oint_\Gamma \frac{\log z dz}{z^{3/4}(1+z)} \tag{4.2}$$

$$\tag{4.3}$$

We can use the same contour as in the previous problem (see figure 1) since we still have a branch cut going from 0 to $\infty$ due to the log, and now there is a simple pole at $z = -1$. Again, we choose the same parametrizations for $\gamma_{in}$ and $\gamma_{out}$ as in problem 1.

$$\left|\int_{\gamma_R} \frac{\log z dz}{z^{3/4}(1+z)}\right| \leq 2\pi R \frac{\log R}{R^{7/4}} \to 0, R \to \infty \tag{4.4}$$

$$\left|\int_{\gamma_\epsilon} \frac{\log z dz}{z^{3/4}(1+z)}\right| \leq 2\pi\epsilon \frac{\log\epsilon}{\epsilon^{3/4}} = 2\pi\epsilon^{1/4}\log\epsilon \to 8\pi\epsilon^{1/4} \to 0, \epsilon \to 0 \tag{4.5}$$

$$\int_{\gamma_{out}} \frac{\log z dz}{z^{3/4}(1+z)} = \int_0^\infty \frac{\log t dt}{t^{3/4}(1+t)} = I \tag{4.6}$$

$$\int_{\gamma_{in}} \frac{\log z dz}{z^{3/4}(1+z)} = -\int_0^\infty \frac{(2\pi i + \log s)ds}{s^{3/4}e^{3\pi i/2}(1+s)} = e^{-3\pi i/2}\left(I + 2\pi i \int_0^\infty \frac{ds}{s^{3/4}(1+s)}\right) \tag{4.7}$$

$$res_{-1} = \frac{\log(-1)}{(-1)^{3/4}} = \pi(1-i)/\sqrt{2} \tag{4.8}$$

$$J = \sqrt{2}\pi^2 i(1-i) = (1-e^{-3\pi i/2})I - 2\pi i e^{-3\pi i/2}\int_0^\infty \frac{dx}{x^{3/4}(1+x)} \tag{4.9}$$

$$\Rightarrow I = -\pi^2\sqrt{2} \tag{4.10}$$

Note that the second integral can be evaluated using methods similar to the first integral.

# 5 Ch. 14, §7, p. 701, 40

$$\int_0^\infty \frac{x dx}{\sinh x} = \frac{1}{2}\int_{-\infty}^\infty \frac{x dx}{\sinh x} \tag{5.1}$$

Consider the contour $\Gamma$ described in the book, a rectangle of height $\pi$ and running from $-R$ to $R$, where we will take $R \to \infty$. The integrals along the ends are bounded above in magnitude by $\frac{R\pi}{\sinh(R)} \to 0$ as $R \to \infty$. Thus, we only need to consider the top and bottom of the rectangle. If we parametrize the top integral by $(x + \pi i), x \in (-R, R)$, we have

$$\oint_\Gamma \frac{z dz}{\sinh z} = \int_{-\infty}^\infty \frac{x dx}{\sinh x} - \int_{-\infty}^\infty \frac{(x+\pi i)dx}{\sinh(x+\pi i)} \tag{5.2}$$

$$= 2\int_{-\infty}^\infty \frac{x dx}{\sinh x} + \int_{-\infty}^\infty \frac{\pi i dx}{\sinh x} = 2\int_{-\infty}^\infty \frac{x dx}{\sinh x} \tag{5.3}$$

$$res_{\pi i} = \left.\frac{z}{\partial_z \sinh z}\right|_{\pi i} = \frac{\pi i}{\cosh \pi i} = -\pi i \tag{5.4}$$

$$\Rightarrow \int_0^\infty \frac{x dx}{\sinh x} = \frac{\pi i}{4}(-\pi i) = \pi^2/4 \tag{5.5}$$

# 6 Ch. 14, §11, p. 720, 27

Following the hint, we will differentiate problem 25 with respect to m.

$$PV\int_0^\infty \frac{\cos mx dx}{x^2-a^2} = -\frac{\pi}{2a}\sin ma \quad (6.1)$$

$$\Rightarrow PV\int_0^\infty \frac{-x\sin mx dx}{x^2-a^2} = -\frac{\pi}{2a}a\cos ma \quad (6.2)$$

$$\Rightarrow PV\int_0^\infty \frac{x\sin mx dx}{x^2-a^2} = \frac{\pi}{2}\cos ma \quad (6.3)$$

# 7 Ch. 14, §11, p.720, 28

$$I = \int_0^\infty \frac{\sqrt{x}\log x dx}{(1+x)^2} \quad (7.1)$$

$$J = \oint_\Gamma \frac{\sqrt{z}\log z dz}{(1+z)^2} \quad (7.2)$$

$$(7.3)$$

We can use the same contour as in problem three (see figure 1) since we still have a branch cut going from 0 to $\infty$ due to the $\sqrt{z}$ and the log, and now there is a pole of order two at $z=-1$. Again, we choose the same parametrizations for $\gamma_{in}$ and $\gamma_{out}$ as in problem 1.

$$\left|\int_{\gamma_R} \frac{\sqrt{z}\log z dz}{(1+z)^2}\right| \leq 2\pi R\frac{R^{1/2}\log R}{R^2} \to 0, R\to\infty \quad (7.4)$$

$$\left|\int_{\gamma_\epsilon} \frac{\sqrt{z}\log z dz}{(1+z)^2}\right| \leq 2\pi\epsilon\sqrt{\epsilon}\log\epsilon \to 0, \epsilon\to 0 \quad (7.5)$$

$$\int_{\gamma_{out}} \frac{\sqrt{z}\log z dz}{(1+z)^2} = \int_0^\infty \frac{\sqrt{t}\log t dt}{(1+t)^2} = I \quad (7.6)$$

$$\int_{\gamma_{in}} \frac{\sqrt{z}\log z dz}{(1+z)^2} = -\int_0^\infty \frac{\sqrt{s}e^{\pi i}(\log s + 2\pi i)ds}{(1+s)^2} = I + 2\pi i\int_0^\infty \frac{\sqrt{s}ds}{(1+s)^2} = I + i\pi^2 \quad (7.7)$$

$$res_{-1} = \pi/2 - i \quad (7.8)$$

$$\Rightarrow J = i\pi^2 + 2\pi = 2I + i\pi^2 \quad (7.9)$$

$$\Rightarrow I = \pi \quad (7.10)$$

# 8 New Problem

## A

The numerator is always finite. So to find the singularities set the denominator to zero:

$$0 = 1 + e^{x+iy} = 1 + e^x(\cos y + i\sin y)$$

Solving the imaginary part we find

$$0 = \sin y$$

so

$$y = \pi N,\ \ N = 0, \pm 1, \pm 2...$$

Solving the real part, we find

$$e^{-x} = -\cos y$$

If $y$ is an odd multiple of $\pi$ then $x = 0$, but if $y$ is an even muliple of $\pi$ then no value of $x$ is a solution. So the final solution set is

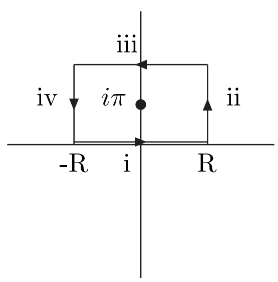
$$x = 0,\ \ y = \pi N,\ \ N = \pm 1 \pm 3...$$

To show that the singularities $z = z_0$ are simple poles, calculate the residues at $z = z_0$ assuming they are simple poles and observe that the result is finite and non-zero.

$$\mathrm{Res}(z_0) = \lim_{z \to z_0} \frac{e^{az}(z - z_0)}{1 + e^z} = e^{az_0} \lim_{z \to z_0} \frac{(z - z_0)}{1 + e^z} = e^{az_0} \frac{1}{e^{z_0}} = e^{a\pi i N}(-1)$$

Since this limit is finite and non-zero for all $N$, the poles are simple and these are indeed the residues.

## B




## C

The only singularity inside the contour is $z = \pi i$, so the residue is $-e^{a\pi i}$ and $I' = -2\pi i e^{a\pi i}$.

**D**

$$
\begin{aligned}
I' &= \lim_{R\to\infty}\left(\int_{-R}^{R} + \int_{R}^{R+2i\pi} + \int_{R+2i\pi}^{-R+2i\pi} + \int_{-R+2i\pi}^{-R}\right)\frac{e^{az}dz}{1+e^{z}} \\
&= \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{e^{ax}dx}{1+e^{x}} + \int_{0}^{2i\pi}\frac{e^{a(R+iy)}idy}{1+e^{(R+iy)}} + \int_{R}^{-R}\frac{e^{a(x+2i\pi)}dx}{1+e^{(x+2i\pi)}} + \int_{2i\pi}^{0}\frac{e^{a(-R+iy)}idy}{1+e^{(-R+iy)}}\right) \\
&= \lim_{R\to\infty}\left(\int_{R}^{-R}\frac{e^{ax}dx}{1+e^{x}} + 0 + e^{2i\pi a}\int_{R}^{-R}\frac{e^{ax}dx}{1+e^{x}} + 0\right) = (1-e^{i2\pi a})I
\end{aligned}
$$

**E**

Equating $I'$ from (C) and (D),

$$-2\pi i e^{a\pi i} = (1-e^{i2\pi a})I$$

Therefore,

$$I = \frac{\pi}{\sin a\pi}$$