Physics 101 Homework 7 Solutions

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1 Ch. 14, §10, 6

Under the conformal map $w = 1/z$, the circle of radius $1/2$ centered at $i/2$ is mapped to the line $w = u - i$, while the line $z = x$ is mapped to $w = u$. Additionally, a point in the upper-half-plane with the disk of radius $1/2$ centered at $i/2$ removed is mapped to the strip $v \in (-1, 0)$. Thus, we want to find the temperature distribution in this strip with $T = 0$ at $v = 0$ and $T = 100$ at $v = -1$. This is given by $T = -100v$. Therefore, since $v = \frac{-y}{x^2+y^2}$, $T = \frac{100y}{x^2+y^2}$ in the upper half plane with the disk removed. The isotherms are given by lines of constant $\frac{y}{x^2+y^2}$ while heat flow is along lines of constant $\frac{x}{x^2+y^2}$, going from the disk to the lower-half-plane.

2 Ch. 14, §10, 9

If we consider the flow in 2 dimensions of an incompressible, non-viscous fluid (aka dry water in 2d), we can use conformal mapping to solve the problem. The velocity potential is constant along lines of constant u, while the streamlines are lines of constant v. Under the map $w = z + 1/z$, $u + iv = x + iy + 1/(x + iy) = x + \frac{x}{x^2+y^2} + i(y - \frac{y}{x^2+y^2})$. Thus, the points A through E are mapped as follows:

$$\texttt{A:} \quad -x \to -\frac{1+x^2}{x}$$

$$\texttt{B:} \quad -1 \to -1 + \frac{1}{-1} = -2$$

$$\texttt{C:} \quad i \to i + \frac{1}{i} = 0$$

$$\texttt{D:} \quad 1 \to 1 + \frac{1}{1} = 2$$

$$\texttt{E:} \quad x \to x + 1/x = \frac{1+x^2}{x}$$

Streamlines are given by lines of constant $y - \frac{y}{x^2+y^2}$, while the velocity potential is constant along lines of constant $x + \frac{x}{x^2+y^2}$.

3 Ch. 14, §10, 12

First show that the mapping $w = \ln\left(\frac{z+1}{z-1}\right)$ maps the boundaries in the w and z planes correctly. For example, we could solve for z, $z = \frac{e^w+1}{e^w-1}$, substitute $w = u + iv$, let $v = \pi/2$, and find the real and imaginary parts x and y in terms of u. This gives $x = \frac{t^2-1}{t^2+1}$ and $y = \frac{-2t}{t^2+1}$, where $t = e^u$. Then we can compute $x^2 + y^2 = 1$, showing that $v = \pi/2$ maps to at least part of the unit circle. Plugging in values of t between 0 and 1 we see that $v = \pi/2$ maps to the bottom half of the unit circle. Repeating the steps for $v = 3\pi/2$ we find that this line maps to the top half of the unit circle.

Alternatively, we can try to map the unit circle to the lines $v = \pi/2$ and $v = 3\pi/2$ with $w = \ln\left(\frac{z+1}{z-1}\right)$. Put the branch cut for the logarithm along the positive reals with $0 < \theta < 2\pi$ (NOTE: Any branch cut for which θ is allowed to vary continuously between $\pi/2$ and $3\pi/2$ works here). Let $z = e^{i\phi}$, find the real and imaginary parts of $\frac{z+1}{z-1}$. Observe that the real part of $\frac{-2i\sin\phi}{2-2\cos\phi}$ is always zero, and that the imaginary part is negative for $0 < \phi < \pi$ and positive for $\pi < \phi < 2\pi$, so that $v = 3\pi/2$ and $v = \pi/2$, respectively, in these ranges. Observe that the magnitude of $\frac{z+1}{z-1}$ goes from zero to infinity for both ranges of ϕ, corresponding to $-\infty < u < \infty$. Therefore the two halves of the circle map onto the entire lines $v = \pi/2$ and $v = 3\pi/2$.

In the w-plane, $T = 10 + \frac{20}{\pi}(v - \pi/2)$. To find T in the z-plane, find v in terms of x and y:

$$v = Im\left(\ln\left(\frac{z+1}{z-1}\right)\right) = Im\left(\ln\left(\frac{x^2+y^2-1-2iy}{(x-1)^2+y^2}\right)\right) = Im\left(\ln\left((...)e^{i\pi + i\arctan\frac{-2y}{x^2+y^2-1}}\right)\right)$$
$$= \pi + \arctan\frac{-2y}{x^2+y^2-1}$$

Note that we add on π since we need to map into $\pi/2$ to $3\pi/2$, not $-\pi/2$ to $\pi/2$.

4 Ch. 14, §10, 13

This problem is identical to §10, 12, with $10 \to V_1$, $30 \to V_2$. So the solution is $V = V_1 + \frac{V_2-V_1}{\pi}(v - \pi/2)$ where $v = \pi + \arctan\frac{-2y}{x^2+y^2-1}$.