Physics 101 Homework 8 Solutions

Michael Johnson, modified by Jason Kaufman

March 1, 2013

1 Ch. 7, §5, 7

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases} \tag{1.1}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{1.2}$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} x dx = \frac{\pi}{2} \tag{1.3}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{1}{\pi}\left(\frac{x}{n}\sin(nx)\Big]_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin(nx)dx\right) \tag{1.4}$$

$$= \frac{\cos(nx)}{\pi n^2}\Big]_0^{\pi} = \begin{cases} -\frac{2}{\pi n^2} & \texttt{n odd} \\ 0 & \texttt{n even} \end{cases} \tag{1.5}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)dx = \frac{1}{\pi}\left(\frac{-x\cos(nx)}{n}\Big]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)dx\right) \tag{1.6}$$

$$= \frac{(-1)^{n+1}}{n} \tag{1.7}$$

$$\Rightarrow f(x) = \frac{\pi}{4} - \frac{2}{\pi}(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...) + (\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...) \tag{1.8}$$

2 Ch. 7, §7, 7

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases} \tag{2.1}$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\pi} x dx = \frac{\pi}{4} \tag{2.2}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{\pi} xe^{-inx}dx \tag{2.3}$$

$$= \frac{1}{2\pi n^2}[(-1)^n(1 + in\pi) - 1] \tag{2.4}$$

Adding the terms $c_n e^{inx} + c_{-n}e^{-inx}$ reduces the expression to the one derived in the previous problem (7.5.7).

3 Ch. 7, §7, 12

$f(x)$ is real so $f(x) = \overline{f(x)}$. Thus, $\sum c_n e^{inx} = \overline{\sum c_n e^{inx}} = \sum \overline{c_n} e^{-inx}$. Since the exponentials in the expansion are orthogonal we may equate coefficients, giving that $\overline{c_n} = c_{-n}$.

4 Ch. 7, §8, 7

$$f(x) = \begin{cases} 0 & -l < x < 0 \\ x & 0 < x < l \end{cases} \tag{4.1}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{1}{l}\int_0^l x dx = \frac{l}{2} \tag{4.2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{l}\int_0^l x\cos\frac{n\pi x}{l}dx \tag{4.3}$$

$$= \frac{1}{l}\left(\frac{lx}{n\pi}\sin\frac{n\pi x}{l}\bigg]_0^l - \int_0^l \frac{l}{n\pi}\sin\frac{n\pi x}{l}dx\right) \tag{4.4}$$

$$= \frac{l}{n^2\pi^2}\cos\frac{n\pi x}{l}\bigg]_0^l = \begin{cases} -\frac{2l}{n^2\pi^2} & \texttt{n odd} \\ 0 & \texttt{n even} \end{cases} \tag{4.5}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_0^l x\sin\frac{n\pi x}{l}dx \tag{4.6}$$

$$= \frac{1}{l}\left(\frac{-lx}{n\pi}\cos\frac{n\pi x}{l}\bigg]_0^l + \int_0^l \frac{l}{n\pi}\cos\frac{n\pi x}{l}dx\right) \tag{4.7}$$

$$= \frac{-l}{n\pi}(-1)^n \tag{4.8}$$

$$c_0 = \frac{1}{2l}\int_0^l x dx = \frac{l}{4} \tag{4.9}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}}dx = \frac{1}{2l}\int_0^l xe^{\frac{-in\pi x}{l}}dx \tag{4.10}$$

$$= \frac{1}{2l}\left(\frac{ilx}{n\pi}e^{\frac{-in\pi x}{l}}\bigg]_0^l - \int_0^l \frac{il}{n\pi}e^{\frac{-in\pi x}{l}}dx\right) \tag{4.11}$$

$$= \frac{il}{2n\pi}(-1)^n + \frac{l}{2n^2\pi^2}((-1)^n - 1) \tag{4.12}$$

$$= \begin{cases} \frac{il}{2n\pi} & \texttt{n even} \\ \frac{-il}{2n\pi} - \frac{l}{n^2\pi^2} & \texttt{n odd} \end{cases} \tag{4.13}$$

$$\tag{4.14}$$

5 Ch. 7, §8, 13

(a)

$$
\begin{aligned}
f(x) &= 2-x, \quad -2<x<2 && (5.1)\\
a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{2} 2-xdx = 4 && (5.2)\\
a_n &= \frac{1}{2}\int_{-2}^{2}(2-x)\cos\frac{n\pi x}{2}dx = \frac{4\sin(n\pi)}{n\pi} = 0 && (5.3)\\
b_n &= \frac{1}{2}\int_{-2}^{2}(2-x)\sin\frac{n\pi x}{2}dx = \left.\frac{-2n\pi(2-x)\cos(\frac{n\pi x}{2})-4\sin(\frac{n\pi x}{2})}{2n^2\pi^2}\right]_{-2}^{2} && (5.4)\\
&= \frac{8n\pi\cos(n\pi)-8\sin(n\pi)}{2n^2\pi^2} = (-1)^n\frac{4}{n\pi} && (5.5)\\
c_0 &= 1/4\int_{-2}^{2}(2-x)dx = \frac{1}{2}a_0 = 2 && (5.6)\\
c_n &= \frac{1}{4}\int_{-2} 2(2-x)e^{-in\pi x/2}dx = \left.\frac{-e^{-in\pi x/2}(4-2in\pi(2-x))}{4n^2\pi^2}\right]_{-2}^{2} && (5.7)\\
&= (-1)^{n+1}\frac{2i}{n\pi} && (5.8)
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(x) &= 2-x, \ \ 0<x<4 && (5.9)\\
a_0 &= \frac{1}{2}\int_{0}^{4} f(x)dx = 0 && (5.10)\\
a_n &= \frac{1}{2}\int_{0}^{4}(2-x)\cos\frac{n\pi x}{2}dx = \frac{-2(-1+\cos(2n\pi)+n\pi\sin(2n\pi))}{n^2\pi^2} = 0 && (5.11)\\
b_n &= \frac{1}{2}\int_{0}^{4}(2-x)\sin\frac{n\pi x}{2}dx = \frac{4\cos(n\pi)(n\pi\cos(n\pi)-\sin(n\pi))}{n^2\pi^2} = \frac{4}{n\pi} && (5.12)\\
c_0 &= 1/4\int_{0}^{4}(2-x)dx = \frac{1}{2}a_0 = 0 && (5.13)\\
c_n &= \frac{1}{4}\int_{0} 4(2-x)e^{-in\pi x/2}dx = \frac{-2ie^{-in\pi}(n\pi\cos(n\pi)-\sin(n\pi))}{n^2\pi^2} = \frac{-2i}{n\pi} && (5.14)
\end{aligned}
$$

6 Ch. 7, §9, 1

(a) $e^{inx} = \cos(nx) + i\sin(nx)$. $\cos nx$ is even and $i\sin(nx)$ is odd.

(b) $xe^x = x\left[\frac{e^x+e^{-x}}{2} + \frac{e^x-e^{-x}}{2}\right] = x[\cosh(x)+\sinh(x)] = x\cosh(x) + x\sinh(x)$. But $\cosh(x)$ is even and $\sinh(x)$ is odd so $x\cosh(x)$ is odd and $x\sinh(x)$ is even.

7 Ch. 7, §9, 23

We'll take the origin to be at the middle of the string, keeping the zero displacement of the string at 0 height. Thus:

$$f(x) = \begin{cases} \frac{2hx}{l} + h & -\frac{l}{2} < x < 0 \\ -\frac{2hx}{l} + h & 0 < x < \frac{l}{2} \end{cases}$$

The function is even so $b_n = 0$ for all n. Also:

$$a_0 = \frac{2}{l}\int_{-l/2}^{l/2} f(x,0)dx = \frac{4}{l}\int_0^{l/2} h - \frac{2hx}{l}dx = h \tag{7.1}$$

$$a_n = \frac{4}{l}\int_0^{l/2}(h - \frac{2hx}{l})\cos(\frac{2n\pi x}{l})dx \tag{7.2}$$

$$= -\frac{4h}{l}\left(\frac{l}{2n\pi}\right)\sin\left(\frac{2n\pi x}{l}\right)\Big]_0^{l/2} - \frac{4}{l}\left(\frac{2h}{l}\right)\left(\frac{l^2(-1+(-1)^n)}{4n^2\pi^2}\right) \tag{7.3}$$

$$= \begin{cases} \frac{4h}{n^2\pi^2} & \texttt{n odd} \\ 0 & \texttt{n even} \end{cases} \tag{7.4}$$

Thus:

$$f(x,0) = \frac{h}{2} + \frac{4h}{\pi^2}\left(\cos\left(\frac{2\pi x}{l}\right) + \frac{1}{9}\cos\left(\frac{6\pi x}{l}\right) + \frac{1}{25}\cos\left(\frac{10\pi x}{l}\right) + ...\right)$$

8 Ch. 7, §12, 3

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ 0 & |x| > \pi \end{cases} \tag{8.1}$$

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx \tag{8.2}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}(-1)e^{-i\alpha x}dx + \frac{1}{2\pi}\int_0^{\pi}(1)e^{-i\alpha x}dx \tag{8.3}$$

$$= \frac{i}{2\pi\alpha}[(e^{-i\alpha\pi} - 1) - (1 - e^{i\alpha\pi})] \tag{8.4}$$

$$= \frac{i}{\pi\alpha}(\cos(\pi\alpha) - 1) \tag{8.5}$$

Now we compute the inverse Fourier transform of $g(\alpha)$ to recover $f(x)$:

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}d\alpha \tag{8.6}$$

$$= \int_{-\infty}^{\infty} \frac{i}{\pi\alpha}(\cos(\pi\alpha) - 1)e^{i\alpha x}d\alpha \tag{8.7}$$

$$= \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha}(\frac{e^{i\pi\alpha} + e^{-i\pi\alpha}}{2} - 1)e^{i\alpha x}d\alpha \tag{8.8}$$

$$= \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha(x+\pi)}}{2\alpha} + \frac{e^{i\alpha(x-\pi)}}{2\alpha} - \frac{e^{i\alpha x}}{\alpha}d\alpha \tag{8.9}$$

See Ch. 7, §12, 8 (the next problem) for a detailed explanation on how to compute these kinds of integrals. Here we will be succinct.

L'Hopital's rule shows that the integrand is non-singular at $\alpha = 0$ so it has no poles. For $x > \pi$, close the contour in the upper half plane (UHP). By Cauchy's theorem, the integral is zero. For $x < -\pi$, close the contour in the lower half plane (LHP). By Cauchy's theorem, the integral is zero.

For $-\pi < x < 0$, deform the contour near $\alpha = 0$ into the LHP. Compute each of the three terms, I_1, I_2, and I_3 separately, by closing the contours in the UHP, LHP, and LHP, respectively. By Cauchy's theorem, $I_2 = I_3 = 0$. By the residue theorem, $I_1 = \frac{i}{\pi}(2\pi i)\frac{1}{2} = -1$. Therefore $f(x) = -1$ in this range.

For $0 < x < \pi$, deform the contour near $\alpha = 0$ into the UHP. Compute each of the three terms, I_1, I_2, and I_3 separately, by closing the contours in the UHP, LHP, and UHP, respectively. By Cauchy's theorem, $I_1 = I_3 = 0$. By the residue theorem, $I_2 = \frac{i}{\pi}(-2\pi i)\frac{1}{2} = 1$. Therefore $f(x) = 1$ in this range.

Note: I somewhat cleverly chose different deformations for the two different x intervals to make the subsequent evaluation of the integrals as efficient as possible. This is not necessary, but depending on how complicated the terms in the integral are, choosing the deformation wisely can save some time.

9 Ch. 7, §12, 8

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{9.1}$$

$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{9.2}$$

$$= \frac{1}{2\pi}\int_0^1 xe^{-ikx}dx \tag{9.3}$$

$$= \frac{i}{2\pi}\frac{d}{dk}\int_0^1 e^{-ikx}dx \tag{9.4}$$

$$= \frac{i}{2\pi}\frac{d}{dk}\left[\frac{ie^{-ikx}}{k}\right]_{x=0}^{x=1} \tag{9.5}$$

$$= \frac{-1}{2\pi}\frac{d}{dk}\left(\frac{e^{-ik}-1}{k}\right) \tag{9.6}$$

$$= \frac{1}{2\pi k^2}\left((ik+1)e^{-ik}-1\right) \tag{9.7}$$

$$\Rightarrow f(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi k^2}\left((ik+1)e^{-ik}-1\right)e^{ikx} \tag{9.8}$$

First, note that the point $k = 0$ is not a pole, since as $k \to 0$ the integrand is finite. This is easily shown by Taylor expanding e^{-ik} or using l'Hopital's rule. For $x > 1$, we can close the contour with a semicircle of infinite radius in the upper half plane. By Cauchy's theorem, the integral is zero. For $x < 0$, we can close the contour in the lower half plane. By Cauchy's theorem, the integral is zero.

For $0 < x < 1$, we cannot close the contour in the lower half or upper half plane since the contribution from the infinite semicircle is infinite rather than zero, so we have to be a bit more creative (or simply follow the advice on the assignment sheet!). Since the integrand is analytic everywhere near the real axis, we can deform the contour slightly to go around the point $k = 0$ without changing the value of the integral. Near $k = 0$ let's move the contour slightly off the real axis into the upper half plane (Note: Since this choice is arbitrary, we could have moved it into the lower half plane). The advantage of avoiding the point $k = 0$ is that we can now integrate each term separately without any of them diverging along the contour.

NOTE: You may ask why we cannot simply integrate through a singular point by taking the principle value. The reason is that taking the principle value is only one way of integrating through a singular point, so unless you have a mathematical or physical reason for choosing this method (or you're asked to do so on a problem set or exam!), you cannot assume that it will give you the answer you're looking for.

The three terms are

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{ie^{ik(x-1)}}{k}$$

,

$$I_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(x-1)}}{k^2}$$

$$I_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-e^{ikx}}{k^2}dk$$

Since $0 < x < 1$, close the contours in the lower half plane for I_1 and I_2, and the upper half plane for I_3. For I_1 and I_2 the contour then encloses $k = 0$ (and note that for I_2, $k = 0$ is a pole of order 2), so $I_1 = \frac{1}{2\pi}(-2\pi i)\, i = 1$ and $I_2 = \frac{1}{2\pi}(-2\pi i)\,(i(x-1)) = x - 1$. For I_3 the contour does not enclose the point $k = 0$ so by Cauchy's theorem, $I_3 = 0$. In total, then, for $0 < x < 1$, $f(x) = I_1 + I_2 + I_3 = x$.

For practice, let's see what would have happened if we had deformed the contour slightly into the lower half plane. Then, as before, we close the contours in the lower half plane for I_1 and I_2, and the upper half plane for I_3. But this time only I_3 encloses the singularity $k = 0$ so $I_1 = I_2 = 0$ and $I_3 = \frac{1}{2\pi}(2\pi i)(-ix) = x$. So, as before, for $0 < x < 1$, $f(x) = I_1 + I_2 + I_3 = x$.

One final comment: Let's denote the integrals for the first method (deforming the contour into the upper half plane) as I_{1U}, I_{2U}, and I_{3U}, and for the second method (deforming the contour into the lower half plane) as I_{1L}, I_{2L}, and I_{3L}. Since $I = I_{1U} + I_{2U} + I_{3U}$ and $I = I_{1L} + I_2L + I_{3L}$, it is also true that $I = \frac{I_{1U}+I_{1L}}{2} + \frac{I_{2U}+I_{2L}}{2} + \frac{I_{3U}+I_{3L}}{2}$. Let's examine $\frac{I_{1U}+I_{1L}}{2}$, for example. This is equal to $\frac{1}{2}(-2\pi i)Res(k = 0)$, since in both cases we close the contour in the lower half plane, but in only one case does the contour enclose the singularity $k = 0$. But this is the same as the formula for the principle value when integrating through a simple pole! In other words, rather than deforming the contour we could instead apply the principle value formula for integrating through a simple pole. Note, however, that even if $k = 0$ were a multiple pole, not a simple pole, we would still be correct to apply this formula here, and in doing so we would NOT actually be evaluating the principle value. For example, $\frac{I_{3U}+I_{3L}}{2} = \frac{1}{2}(2\pi i)Res(k = 0)$ but the principle value of I_3 is actually infinite. If this all seems a bit convoluted, then stick to deforming the contour for now, but make sure you revisit this and understand the details before the end of the course.

10 Ch. 7, §12, 21

$$\begin{aligned}
f(x) &= e^{-x^2/(2\sigma^2)} && (10.1)\\
g(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)}e^{-ikx}dx && (10.2)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x+i\sigma^2 k)^2/(2\sigma^2)}e^{-\sigma^2k^2/2}dx && (10.3)\\
&= \frac{e^{-\sigma^2k^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{-(x+i\sigma^2 k)^2/(2\sigma^2)}dx && (10.4)\\
&= \frac{\sigma e^{-\sigma^2k^2/2}}{\sqrt{2\pi}} && (10.5)\\
\Rightarrow f(x) &= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\sigma^2k^2/2}e^{ikx}dk && (10.6)\\
&= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-\sigma^2}{2}(k^2+\frac{2ix}{\sigma^2}k)}dk && (10.7)\\
&= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-\sigma^2}{2}[(k+\frac{ix}{\sigma^2})^2+\frac{x^2}{\sigma^4}]}dk && (10.8)\\
&= \frac{\sigma e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-\sigma^2}{2}(k+\frac{ix}{\sigma^2})^2}dk && (10.9)\\
&= \frac{\sigma e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}}\sqrt{\frac{\pi}{\sigma^2/2}} && (10.10)\\
&= e^{-x^2/(2\sigma^2)} && (10.11)
\end{aligned}$$

11 Ch. 7, §13, 4

a)

$$\begin{aligned}
R\frac{dq}{dt}+\frac{q}{C} &= V && (11.1)\\
\Rightarrow \frac{dq}{dt} &= \frac{V}{R}-\frac{q}{RC}=\frac{CV-q}{RC} && (11.2)\\
\Rightarrow \int_o^q \frac{RC\,dq'}{CV-q'} &= T && (11.3)\\
\Rightarrow RC\ln\left(\frac{CV}{CV-q}\right) &= T && (11.4)\\
\Rightarrow q &= CV\left(1-e^{-\frac{T}{RC}}\right) && (11.5)
\end{aligned}$$

c) Our period is $\frac{1}{2}RC$.

$$
\begin{aligned}
a_0 &= \frac{4}{RC}\int_0^{\frac{1}{2}RC} CV\left(1-e^{-t/RC}\right)dt = \frac{4}{RC}\left(\frac{RC^2V}{2} + RC^2V(-1+\frac{1}{\sqrt{l}})\right) && (11.6)\\
&= 2CV\left(\frac{2}{\sqrt{l}}-1\right) && (11.7)\\
a_n &= \frac{4}{RC}\int_0^{1/2RC} CV\left(1-e^{-t/RC}\right)\cos\left(\frac{4n\pi t}{RC}\right)dt = \frac{4C(1-\sqrt{l})V}{\sqrt{l}(1+16n^2\pi^2)} && (11.8)\\
b_n &= \frac{4}{RC}\int_0^{1/2RC} CV\left(1-e^{-t/RC}\right)\sin\left(\frac{4n\pi t}{RC}\right)dt = \frac{4C(4n\pi-4\sqrt{l}n\pi)V}{\sqrt{l}(1+16n^2\pi^2)} && (11.9)
\end{aligned}
$$

where l is just e, the base of the natural log.