Physics 101 Homework 9 Solutions

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1 New Problem

We are solving

$$\ddot{x} + 2\alpha\dot{x} + \omega_o^2 x = A(t)$$

In class we showed that $x(t)$ is the convolution of $G(t)$ and $A(t)$, where $G(t)$ is the solution for $A(t) = \delta(t)$. That is,

$$x(t) = \int_{-\infty}^{\infty} G(t-\tau)A(\tau)d\tau$$

Define $\Omega = \sqrt{\omega_0^2 - \alpha^2}$. For the underdamped case, i.e. $\alpha < \omega_o$, $G(t) = 0$ for $t < 0$ and $G(t) = \frac{e^{-\alpha t}}{\Omega}\sin(\Omega t)$ for $t \geq 0$.

$$x(t) = \int_{-\infty}^{\infty} G(t-\tau)A(\tau)d\tau = \int_0^T G(t-\tau)A_0 d\tau$$

since $A(t) = 0$ for $t < 0$ and $t > T$. Now, since $G(t-\tau) = 0$ for $\tau > t$,

$$x(t) = \int_0^{\min(T,t)} G(t-\tau)A_0 d\tau = \int_0^{\min(T,t)} \frac{e^{-\alpha(t-\tau)}}{\Omega}\sin(\Omega(t-\tau))A_0 d\tau$$

Writing the sine function in terms of exponentials and integrating (or integrating by parts), we get

$$\begin{aligned}
x(t) &= \left(\frac{A_0}{\Omega}\right)\left(\frac{1}{\alpha^2+\Omega^2}\right)\left(e^{\alpha(\tau-t)}(\Omega\cos(\Omega(t-\tau)) + \alpha\sin(\Omega(t-\tau)))\right)\Big|_0^{\min(t,T)} \\
&= \left(\frac{A_0}{\Omega}\right)\left(\frac{1}{\alpha^2+\Omega^2}\right)\left(\Omega - e^{-\alpha t}(\Omega\cos\Omega t + \alpha\sin\Omega t)\right),\ t < T \\
&= \left(\frac{A_0}{\Omega}\right)\left(\frac{1}{\alpha^2+\Omega^2}\right)\left(e^{\alpha(T-t)}\left(\Omega\cos\Omega(T-t) - \alpha\sin\Omega(T-t)\right) - e^{-\alpha t}(\Omega\cos\Omega t + \alpha\sin\Omega t)\right),\ t \geq T
\end{aligned}$$

For the overdamped case, i.e. for $\alpha > \omega_o$, it is easiest to note that Ω is imaginary and then use the following identities and substitutions $\Omega = i\Gamma$, $\sin i\theta = i\sinh\theta$, $\cos i\theta = \cosh\theta$.

$$\begin{aligned}
x(t) &= \left(\frac{A_0}{\Gamma}\right)\left(\frac{1}{\alpha^2-\Gamma^2}\right)\left(\Gamma - e^{-\alpha t}(\Gamma\cos\Gamma t + \alpha\sin\Gamma t)\right),\ t < T \\
&= \left(\frac{A_0}{\Gamma}\right)\left(\frac{1}{\alpha^2+\Gamma^2}\right)\left(e^{\alpha(T-t)}\left(\Gamma\cos\Gamma(T-t) - \alpha\sin\Gamma(T-t)\right) - e^{-\alpha t}(\Gamma\cos\Gamma t + \alpha\sin\Gamma t)\right),\ t \geq T
\end{aligned}$$

The critically damped case is solved in the lecture notes.

2 Ch. 8, §9, 18

$$\begin{aligned}
y'' - 4y &= 3e^{-t} && (2.1)\\
\Rightarrow p^2Y - py_0 - y_0' - 4Y &= 3\int_0^\infty e^{-(p+1)t}dt && (2.2)\\
\Rightarrow (p^2-4)Y - p + 3 &= \frac{3}{p+1} && (2.3)\\
\Rightarrow Y &= \frac{p}{(p+1)(p+2)} && (2.4)\\
\Rightarrow y &= \left.\frac{pe^{pt}}{p+2}\right|_{p=-1} + \left.\frac{pe^{pt}}{p+1}\right|_{p=-2} && (2.5)\\
&= 2e^{-2t} - e^{-t} && (2.6)
\end{aligned}$$

3 Ch. 8, §9, 28

$$\begin{aligned}
y' + z = 2\cos t &\qquad z' - y = 1 && (3.1)\\
-y_0 + sY + Z = \int_0^\infty (e^{it}+e^{-it})e^{-st}dt &\qquad -z_0 + sZ - Y = \int_0^\infty e^{-st}dt && (3.2)\\
1 + sY + Z = \frac{2s}{(s+i)(s-i)} &\qquad -s + s^2Z - sY = 1 && (3.3)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 1 - s + s^2Z + Z &= 1 + \frac{2s}{(s+i)(s-i)} && (3.4)\\
\Rightarrow Z &= \frac{2s}{(s+i)^2(s-i)^2} + \frac{s}{(s+i)(s-i)} && (3.5)\\
\Rightarrow z &= 2\Re\left[\frac{(2st+2)e^{st}}{(s-i)^2} - \frac{4se^{st}}{(s-i)^3} + \frac{se^{st}}{s-i}\right]_{s=-i} && (3.6)\\
&= \cos t + t\sin t && (3.7)\\
\Rightarrow 1 + sY + \frac{2s}{(s+i)^2(s-i)^2} + \frac{1}{(s+i)(s-i)} &= \frac{2s}{(s+i)(s-i)} && (3.8)\\
\Rightarrow Y &= \frac{-1}{s} - \frac{2}{(s+i)^2(s-i)^2} + \frac{1}{(s+i)(s-i)} && (3.9)\\
\Rightarrow y &= \Re\left((t-i)e^{-it} + ie^{-it}\right) - 1 && (3.10)\\
&= t\cos t - 1 && (3.11)
\end{aligned}$$

4 Ch. 8, §10, 14

$$
\begin{aligned}
y'' + 5y' + 6y &= e^{-2t} && (4.1)\\
\Rightarrow -y_0' - sy_0 + s^2Y + 5(-y_0 + sY) + 6Y &= L(e^{-2t}) && (4.2)\\
\Rightarrow (6 + s^2 + 5s)Y &= L(e^{-2t}) && (4.3)\\
\Rightarrow Y &= \frac{L(e^{-2t})}{(s+3)(s+2)} && (4.4)\\
&= L(g)L(e^{-2t}) && (4.5)\\
\Rightarrow g &= e^{-2t} - e^{-3t} && (4.6)\\
\Rightarrow Y &= L(e^{-2t})L(e^{-2t} - e^{-3t}) && (4.7)\\
\Rightarrow y &= \int_0^t e^{-2\tau}(e^{-2(t-\tau)} - e^{-3(t-\tau)})d\tau && (4.8)\\
&= te^{-2t} - e^{-2t} + e^{-3t} && (4.9)
\end{aligned}
$$

5 Ch. 8, §10, 18

$$
\begin{aligned}
y'' + \omega^2 y &= f && (5.1)\\
\Rightarrow -y_0' - sy_0 + s^2Y + \omega^2 Y &= F && (5.2)\\
\Rightarrow Y &= \frac{F}{(s+i\omega)(s-i\omega)} && (5.3)\\
&= FG && (5.4)\\
\Rightarrow g &= \frac{e^{i\omega t}}{2i\omega} + \frac{e^{-i\omega t}}{-2i\omega} && (5.5)\\
&= \omega^{-1}\sin\omega t && (5.6)\\
\Rightarrow Y &= L(\frac{\sin\omega t}{\omega})F && (5.7)\\
\Rightarrow y &= \int_0^t \frac{\sin(\omega(t-\tau))f(\tau)}{\omega}d\tau && (5.8)\\
&= \begin{cases} \frac{1-\cos\omega t}{\omega^2} & t < a \\ \frac{\cos(\omega(t-a))-\cos\omega t}{\omega^2} & t > a \end{cases} && (5.9)
\end{aligned}
$$

6 Ch. 14, §7, 63

$$\begin{aligned} Y &= \frac{p}{p^4-1} && (6.1)\\ &= \frac{p}{(p+1)(p-1)(p+i)(p-i)} && (6.2)\\ \Rightarrow y &= \frac{e^t}{4} + e^{-t}4 + \frac{ie^{it}}{-4i} + \frac{-ie^{-it}}{4i} && (6.3)\\ &= \frac{\cosh t - \cos t}{2} && (6.4) \end{aligned}$$

7 Ch. 14, §7, 66

We want to show u even and v odd $\Leftrightarrow f^*(x) = f(-x)$. Consider first the forward direction. $f(x) = u(x) + iv(x)$, so

$$f^*(x) = u(x) - iv(x) = u(-x) + iv(-x) = f(-x) \tag{7.1}$$

Now to consider the other direction, we have $f^*(x) = f(-x)$. Thus,

$$u(x) = \frac{f(x)+f^*(x)}{2} = \frac{f(x)+f(-x)}{2} \tag{7.2}$$

but this is just the even part of $f(x)$. Similarly,

$$v(x) = \frac{f(x)-f^*(x)}{2i} = \frac{f(x)-f(-x)}{2i} \tag{7.3}$$

which is clearly an odd function of x.

$$\pi u(a) = PV \int_{-\infty}^{\infty} \frac{v(x)}{x-a} dx \quad (7.4)$$

$$= PV \left(\int_{-\infty}^{0} \frac{v(x)}{x-a} dx + \int_{0}^{\infty} \frac{v(x)}{x-a} dx \right) \quad (7.5)$$

$$= PV \left(\int_{\infty}^{0} \frac{v(-x)}{-x-a} (-dx) + \int_{0}^{\infty} \frac{v(x)}{x-a} dx \right) \quad (7.6)$$

$$= PV \int_{0}^{\infty} \left(\frac{v(x)}{x+a} + \frac{v(x)}{x-a} \right) dx \quad (7.7)$$

$$= PV \int_{0}^{\infty} \frac{(x-a+x+a)v(x)}{x^2-a^2} dx \quad (7.8)$$

$$= PV \int_{0}^{\infty} \frac{2xv(x)}{x^2-a^2} dx \quad (7.9)$$

$$\pi v(a) = -PV \int_{-\infty}^{\infty} \frac{u(x)}{x-a} dx \quad (7.10)$$

$$= -PV \left(\int_{-\infty}^{0} \frac{u(x)}{x-a} dx + \int_{0}^{\infty} \frac{u(x)}{x-a} dx \right) \quad (7.11)$$

$$= -PV \left(\int_{\infty}^{0} \frac{u(-x)}{-x-a} (-dx) + \int_{0}^{\infty} \frac{u(x)}{x-a} dx \right) \quad (7.12)$$

$$= -PV \int_{0}^{\infty} \left(\frac{-u(x)}{x+a} + \frac{u(x)}{x-a} \right) dx \quad (7.13)$$

$$= -PV \int_{0}^{\infty} \frac{(a-x+x+a)u(x)}{x^2-a^2} dx \quad (7.14)$$

$$= -PV \int_{0}^{\infty} \frac{2au(x)}{x^2-a^2} dx \quad (7.15)$$

8 Ch. 8, §12, 15

Note: In solving differential equations with Green's functions, Boas implies that the inhomogeneous term is assumed to be zero before some particular value, usually $t = 0$ (or $x = 0$). This corresponds, for example, to a system responding to a force that is zero until a particular time. We will make this assumption with the following two problems, so that we can solve them with retarded Green's functions. Otherwise, the solution will diverge, as we will demonstrate below.

The retarded Green's function is:

$$G = (A \sinh x + B \cosh x)\theta(x - x') = f(x)\theta(x - x')$$

$$\Rightarrow G' = f'(x)\theta(x - x') + f(x)\delta(x - x') = f'(x)\theta(x - x') + f(x')\delta(x - x')$$

$$\Rightarrow G'' = f''(x)\theta(x-x') + f'(x)\delta(x-x') + f(x')\delta'(x-x') = f(x)\theta(x-x') + f'(x')\delta(x-x') + f(x')\delta'(x-x')$$

Where we've used that f"(x)=f(x). Now, we want that $G'' - G = \delta(x - x')$ so plugging in the above and equating coefficients yields:

$$f'(x') = A \cosh(x') + B \sinh(x') = 1$$

$$f''(x') = A\sinh(x') + B\cosh(x') = 0$$

$$\Rightarrow A = \cosh x', \;\; B = -\sinh x'$$

Our integral formula with the Green's function now gives:

$$y = \int_0^x \frac{(\cosh x' \sinh x - \sinh x' \cosh x)}{\cosh x'} dx'$$

$$\Rightarrow y = \cosh(x)\left[x + \ln\left(\frac{2}{1+e^{2x}}\right)\right] + x\sinh x = x\sinh x - \cosh x \ln(\cosh x)$$

for $x > 0$, and $y = 0$ otherwise.

Note: The lower bound on the integral is zero because, as mentioned above, we are assuming that the inhomogeneous term is 0 for $x < 0$ (i.e. it is a piecewise function). Otherwise, the solution would diverge. Of course, one could analytically continue the $x > 0$ solution to $x < 0$, but this is just a clever solution to a *different* problem. Namely, this would be a problem where the inhomogeneous term is always nonzero, and a boundary condition is $y(0) = 0$. To solve this formally using Green's functions (rather than cleverly analytically continuing our answer from above), one would have to start with a more general Green's function, as explained in Lecture 22.

9 Ch. 8, §12, 16

Again, let's use a retarded Green's function (see previous problem for caveats).

$$G = (Ax + Bx^2)\theta(x - x')$$

Taking derivatives gives:

$$G' = (A + 2Bx)\theta(x - x') + (Ax + Bx^2)\delta(x' - x) = (A + 2Bx)\theta(x - x') + (Ax' + B(x')^2)\delta(x - x')$$

$$G'' = 2B\theta(x - x') + (A + 2Bx')\delta(x - x') + (Ax' + B(x')^2)\delta'(x - x')$$

Now, since $x^2G'' - 2xG' + 2G = \delta(x - x')$, we may plug in our above expressions and equation coefficients. The θ terms drop out and the coefficients of δ and δ' give (respectively):

$$A(x')^2 - 2B(x')^3 - 2A(x')^2 + 2B(x')^3 = 1 \Rightarrow A(x')^2 = -1$$

$$Ax' + B(x')^2 = 0 \Rightarrow A + Bx' = 0$$

Thus, $A = -(x')^{-2}$, and $B = (x')^{-3}$. This time, if we assume that the inhomogeneous term is "turned on" at $x = 0$, the solution still diverges. So let's turn it on at $x = 1$. This gives:

$$y = \int_1^x x' \ln x' \left(-\frac{x}{(x')^2} + \frac{x^2}{(x')^3}\right) dx' = (x-1)x - \frac{1}{2}x\ln x(2 + \ln x)$$

for $x > 1$, and $y = 0$ otherwise.

You may easily check that this is a solution to $x^2y'' - 2xy' + 2y = x\ln x$.