Phys 110C: Problems for HW 3

C. Gwinn

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1 HW3 1: Group Velocity and Phase Velocity

The speed that appears in the wave equation is sometimes called "phase velocity" v_ϕ. It is the speed of a surface (or, in 1D a point) of constant phase, in a sinusoidal wave. If the wave is written $E(z) = E_0 \exp\{i(kz - \omega t)\}$, then $v_\phi = \omega/k$, where ω is the wave frequency and k is the wavenumber.

Sometimes the wave frequency, and the phase velocity depend on the wavenumber k. This isn't the case for light waves in vacuum; but it *is* the case for light waves in materials, or for de Broglie waves for particles with $v << c$. In these cases, it's useful to think about the speed at which a signal travels through the medium. This travels at the group velocity v_g. The group velocity can be quite different from the phase velocity; it can even be in another direction. (Look at a boat's wake to see an example of this.) In this problem, you will show that (in 1D) the group velocity is $v_g = \partial\omega(k)/\partial k$.

Consider a *wave packet*, traveling toward $+\hat{z}$ through a medium with wave frequency $\omega(k)$. You can visualize this as a wave of a particular frequency, but with spatial extent limited by a Gaussian function. The Gaussian function is centered at z_0 and its standard deviation is σ. At time $t = 0$, the electric field is given by:

$$E(z) = \tilde{E}_0 \exp\{i(k_0 z)\} \exp\left\{-\frac{1}{2}\frac{(z - z_0)^2}{\sigma^2}\right\}. \quad (1)$$

We want to find how the wavepacket moves with time.

a) First, prove a lemma. Complete the square to show that:

$$\int_{-\infty}^{\infty} du\, e^{-\frac{1}{2}u^2 + bu} = e^{\frac{1}{2}b^2} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2} = e^{\frac{1}{2}b^2}\sqrt{2\pi}. \quad (2)$$

where $x = u - b$. Note that this fact holds whether b is real, imaginary, or complex.

b) We can express the wave either as a truncated sinusoid as in Eq. 1, or as a superposition of pure sinusoids. Show that the wave packet can be expressed in the form:

$$E(z) = \int_{-\infty}^{\infty} dk \tilde{A}(k) \exp\{i(k(z - z_0))\} \tag{3}$$

where

$$\tilde{A}(k) = \tilde{E}_0 \frac{\sigma}{\sqrt{2\pi}} e^{ik_0 z_0} \exp\left\{-\frac{1}{2}(k - k_0)^2 \sigma^2\right\}. \tag{4}$$

(Hint: Plug Eq. 4 into Eq. 3, and use the fact from part a).

Eq. 3 represents the wave in Fourier space. Eq. 4 shows that the finite extent of the wave packet in real space corresponds to a finite range of wavenumbers in Fourier space, as given by the uncertainty principle: $\Delta z \cdot \Delta k \approx 1$

c) The time evolution is hard to determine in real space. But, in Fourier space, each k evolves as a plane wave, with frequency $\omega(k)$. In fact,

$$E(z, t) = \int_{-\infty}^{\infty} dk \tilde{A}(k) \exp\{i(k\ (z - z_0) - \omega(k)t)\}. \tag{5}$$

As a simple model, suppose that $\omega(k) = \omega_0 + v_g(k - k_0)$, where v_g is a constant. (The two terms can be regarded as the first two terms of a Taylor series for $\omega(k)$. The approximation is best when the frequency range is narrow; that is, when $\sigma >> 2\pi/k$.) This simple model provides a (somewhat ungainly) expression for $E(z, t)$.

You can simplify the expression by defining the function:

$$z_1(t) = z_0 + v_g t, \tag{6}$$

and the lumped constant $\omega_1 = \omega_0 - v_g k_0$. This yields:

$$E(z, t) = \int_{-\infty}^{\infty} dk \tilde{A}(k) \exp\{i(k(z - z_1) - \omega_1 t)\}. \tag{7}$$

What is the effect of the time evolution on the wave packet in Fourier space?

d) Finally, show that

$$E(z,t) = \tilde{E}_0 \exp\{i(k_0 z - \omega_1 t)\} \exp\left\{-\frac{1}{2}\frac{(z - z_1(t))^2}{\sigma^2}\right\}. \tag{8}$$

Here, you can complete the square again, or just use your results from part b backward, with an offset for z and an additional phase factor.

Argue that this can be interpreted as a wave packet traveling at velocity v_g. The packet is composed of waves traveling at the phase velocity $v_p = \omega_1/k_0$.

For fun, you can continue the Taylor series for $\omega(k)$ to the quadratic term, and find how rapidly the wave packet broadens as it travels, using part a with a complex b.

2 HW3 2: Examples of Group Velocity and Phase Velocity

This problem is based on 9-20 in Griffiths.

a) The dispersion relation for a water wave, in water of depth d, is

$$\omega = \sqrt{(gk)\tanh(kd)}. \tag{9}$$

where g is the acceleration of gravity. Here we consider two limits: "deep" water ($d >> \lambda = 2\pi/k$) and "shallow" water ($d << \lambda = 2\pi/k$).

Find the phase velocity of waves in shallow water, and show that the waves are nondispersive. Show that group velocity is equal to phase velocity.

Find the phase velocity in deep water.

Find the group velocity in deep water.

Note that the same water may be deep to one wave and shallow to another, depending on wavelength. Around here, a typical swell period might be 14 sec. Find the water depth at which a wave of this period makes the transition from deep to shallow. Wave breaking is a related phenomenon, said to occur when water depth is comparable to wave height.

b) De Broglie waves (matter waves) have frequency $\omega = E/\hbar$ given by energy E, and wavenumber $k = p/\hbar$ given by momentum p.

Find the group and phase velocity for De Broglie waves, for non-relativistic particles.

Which velocity corresponds to particle velocity v?

3 HW3 3: Energy Density inside a conductor

Find the ratio of the time average of magnetic energy density, to the time average of electric energy density, $\langle W_m \rangle / \langle W_e \rangle$, for an electromagnetic wave inside a conductor. Then find approximate expressons for this ratio for the limiting cases of an insulator and a very good conductor.

Find the Poynting vector for a wave inside a conductor. At what wave phase is the rate of energy flow maximum? Is energy conserved, in this case?

4 HW3 4: Refraction at a plasma surface

Find the index of refraction of a plasma with electron density n. An electromagnetic wave of frequency $\omega = 2\omega_p$ travels through vacuum until it is incident on the surface of the plasma, at an angle of incidence θ. Find the angle of refraction. Find the angle of incidence for which total "internal" reflection takes place.

Note that, for a plasma, total "internal" reflection takes place for electromagnetic waves shining from vacuum onto the plasma; whereas usually we think of total "internal" reflection as taking place for light within the material shining onto a surface to vacuum. Why should this be the case?

X-ray telescopes use this principle. Metals have plasma frequencies in the ultraviolet, so that X-rays incident at a shallow angle are 100% reflected. The requirement for a shallow angle demands that the mirrors use large, extremely accurate surfaces nearly parallel to the line of sight. Furthermore, high-energy X-rays demand shallow angles and special materials, or both. But higher-energy X-rays are also rare, so ultimately other, non-imaging technologies become more efficient.

For fun: Can you find a material that would show total "internal" reflection, like the plasma, for visible light from outside?

5 HW3 5: Z-Pinch

This problem investigates the magnetic and pressure forces acting on a current through a plasma, where the magnetic forces are created by currents within the plasma. An Austrian con man, Ronald Richter, convinced the Argentinian dictator Juan Peron that he could produce fusion energy from such devices. A steadily increasing current through a plasma, he reasoned, would create a surrounding, confining magnetic field that would compress the plasma until densities and pressures were right for a self-sustaining fusion reaction. Richter's work stimulated work on magnetism in Argentina, and helped to ignite the imagination of US and Russian scientists, who later invented stellarators and tokamaks.

a) Consider an equilibrium configuration of current density and magnetic field, confining a plasma. Because the configuration is an equilibrium, $\frac{\partial}{\partial t} = 0$. The net electromagnetic force per volume on a group of streaming electrons (or ions) is given by the Lorentz force law. This force must be countered by pressure p. Combine these facts to show that, in equilibrium,

$$\vec{\nabla} p = \vec{J} \times \vec{B}. \tag{10}$$

Show that you can combine this with Faraday's Law (with $\frac{\partial}{\partial t} = 0$) and Gauss's Law for magnetism to show that

$$\vec{\nabla}\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}(\vec{B} \cdot \vec{\nabla})\vec{B}. \tag{11}$$

b) Consider an axisymmetric current flowing toward $+z$. Thus, the current density takes the form (in cylindrical coordinates) $\vec{J} = J_s(s, z)\hat{s} + J_z(s, z)\hat{z}$. Again, assume equilibrium: $\frac{\partial}{\partial t} = 0$.

Show that $\vec{B} \cdot \hat{z} = 0$. Also show that $\vec{B} \cdot \hat{s} = 0$.

Show that

$$\frac{\partial}{\partial z}\left(p + \frac{B_\phi^2}{2\mu_0}\right) = 0. \tag{12}$$

so that $\left(p + \frac{B_\phi^2}{2\mu_0}\right)$ depends only on s.

Show that

$$\frac{\partial}{\partial s}\left(p + \frac{B_\phi^2}{2\mu_0}\right) = -\frac{B_\phi^2}{\mu_0 s}. \tag{13}$$

Using this equation and your result above, show that p can have no z-dependence. Thus, although equilibria exist for infinitely long currents, no equilibrium can exist for a current that is finite in z.

6 Problems from Griffiths

9.19, 9.20