Phys 110C: Problems for HW 4

C. Gwinn

Due 28 Apr 2009

1 HW4 1: Two-Stream Instability

1.1 Eulerian Description and Convective Derivative

a) Motion of a particle, or collection of particles, can be described in different ways. In the *Lagrangian* description of particle motion, it travels along a curve $\vec{r}(t)$, and its acceleration is given by Newton's 2nd Law:

$$m\,\dot{\vec{v}} = m\,\ddot{\vec{r}(t)} = \vec{F}, \tag{1}$$

where $\vec{F}$ is the force. In an *Eulerian* description (usually applied to a fluid, or a large collection of particles), a vector field $\vec{v}(\vec{r}, t)$ gives the velocity. Newton's 2nd Law takes the form (often called the momentum equation):

$$\frac{m}{\Delta V}\left(\frac{\partial \vec{v}}{\partial t} + \sum_{i=1,3}\left(\frac{\partial}{\partial r_i}\vec{v}\right)\cdot\frac{\partial r_i}{\partial t}\right) = \frac{\vec{F}}{\Delta V} \tag{2}$$

$$\text{or,}\qquad \rho_\ell\left(\frac{\partial \vec{v}_\ell}{\partial t} + (\vec{v}_\ell\cdot\vec{\nabla})\vec{v}_\ell\right) = \vec{f}_\ell \tag{3}$$

where ρ_ℓ is the mass per unit volume and $\vec{f}_\ell$ is the force per unit volume for "species" ℓ. Of course, this force may be different for different particle species: for example, the electric force for negative electrons and positive ions; and the motions can be different as well, as in the case of plasma oscillations. So, we treat each species with a separate momentum equation.

The second term on the left-hand side of the momentum equation is called the convective derivative. It represents acceleration along the path of a particle: for example, water falling out of a tap is subject to gravity and accelerates accordingly, but the velocity at a given height is constant in time. Likewise, electrons may experience acceleration when they leave a cathode; but the velocities of electrons at a given distance from the cathode may be constant in time (see Problem 2.48).

Consider a plasma with equal, static densities of electrons and ions: $n_{e0} = n_{i0} \equiv n_0$. It has no electric field: $E_0 = 0$, and the ions are at rest: $v_{i0} = 0$. However, the electrons travel toward $+x$ with speed v_0. In these expressions, the subscript "0" indicates the zero-order, static case. • Verify that this situation trivially satisfies the momentum equation.

1.2 Perturbative Solution

b) A small, longitudinal electric field is introduced: $\vec{E}(\vec{r}, t) = E_1 \hat{x} \exp\{i(kx - \omega t)\}$. This will produce forces on electrons and ions, resulting in displacements x_{e1}, x_{i1} and velocities v_{e1}, v_{i1}; these represent small (or "first-order") departures from the original state, indicated by the subscript "1". For these quantities, we can use Fourier's trick to assert the same time and space dependence as the electric field, possibly with a phase factor in the coefficient, *so long as the equations are linear.*

Of course, the momentum equation (Eq. 3) is nonlinear; however, as long as the perturbations are small, we can ignore products of 1st-order terms in favor of first-order terms alone and products of zero-order terms with first-order terms. This procedure will linearize the momentum equation, and all the other equations as well. This trick usually works; occasionally it produces nonsense, in which case it's worthwhile looking into your assumptions.

• Show that the momentum equation yields, when linearized to first order:

$$v_{i1} = \frac{ie}{M_i \omega} E_1 \tag{4}$$

$$v_{e1} = -\frac{ie}{m_e} \frac{E_1}{\omega - k v_0}. \tag{5}$$

The equation of continuity is

$$\frac{\partial n_\ell}{\partial t} + \vec{\nabla} \cdot (n_\ell \vec{v}_\ell) = 0, \tag{6}$$

where ℓ denotes the species. (This equation is familiar for conservation of charge; of course, it also holds for conservation of electrons and of ions, as here.) • Show that this yields, to first order:

$$-i\omega n_{i1} + ikn_0 v_{i1} = 0 \tag{7}$$

$$-i\omega n_{e1} + ikv_0 n_{e1} + ikn_0 v_{e1} = 0. \tag{8}$$

• Finally, use Gauss's Law to show that

$$ik\epsilon_0 E_1 \;=\; e(n_{i1} - n_{e1}). \tag{9}$$

1.3 Dispersion Relation

c) • Combine Equations 4 through 9 to find the dispersion relation

$$1 \;=\; \omega_p^2 \left(\frac{m_e/M_i}{\omega^2} + \frac{1}{(\omega - kv_0)^2} \right) \tag{10}$$

• Show that if $v_0 = 0$, then oscillations are at nearly the plasma frequency, with a slight correction for the recoil of the ions. The wavevector k is not determined; it can be anything. These are indeed plasma oscillations, as also found from the wave equation assuming longitudinal waves.

• Show that if we assume that the ions do not recoil ($m_e/M_i \to 0$), and that the electrons travel at speed v_0, then we obtain two solutions for $\omega(k)$. • What do these solutions look like, physically? • Why is there now a definite relation between ω and k?

For fixed k, the dispersion relation is 4th order in ω, so we expect 4 roots for $\omega(k)$. The roots may be real or complex. Because the coefficients of the polynomial are real, any complex roots must come in pairs, which take the form $\omega(k) = \alpha \pm i\beta$, where α, β are real. The root with positive imaginary part will grow exponentially with time, and the other will die. Unless the initial conditions are set very carefully, the unstably growing mode will come to dominate the system.

• Plot the dispersion relation in ω, k space, by sketching or using Mathematica or a similar package. (Caution: You may see some difficulty for graphing routines in Mathematica with a realistic inverse mass-ratio $m_e/M_i \approx 1/1800$. You may wish to get a feel for the equations with a smaller inverse mass-ratio, say 1/20.) One approach is to take $x = \omega$, $y = kv_0$, and 3D plot the left-hand side of Eq. 10 as a function of x, y, with the plot maximum set to 1.

• Using your understanding of the dispersion relation, argue that the imaginary roots (and the instability) lie in the range of ω between points where the group velocity becomes infinite.

• Find the range of instability in terms of v_0 and $\mu \equiv M_i/m_e$. (Hint: You may find this easiest to determine as an expression involving ω, even though ω is technically the dependent variable).

Why does the instability grow? Why is ion motion important? You may wish to do some reading on this.

This instability is known as the two-stream instability, or sometimes the Buneman instability. In our simple model, the instability grows most quickly for $k \to 0$. In practice, the instability can be stabilized by thermal effects (most famously by Landau damping) if the beam velocity is less than a few thermal velocities: $v_0 \lesssim \sqrt{k_B T/m_e}$. However, the instability is one of many that can affect z-pinches (see HW3).

2 HW4 2: Index of Refraction

a) Consider a material with a dielectric constant of the form of Eq. 9.161. Griffiths writes that for gases, where N is small, we can use the binomial expansion to express $\tilde{k}$ as 9.169. Of course, we actually require that a dimensionless constant involving a factor of N be small.

What is that dimensionless constant? For what range of frequencies can we use the binomial expansion for the Lyman-α transition, with $\lambda_\alpha = 121$ nm, oscillator strength $f_\alpha = 0.416$, and $\gamma_\alpha = 10^{-7}\omega_\alpha$, given a characteristic interstellar density of $N_H = 1\ \text{cm}^{-3}$?

b) Consider a layer of material with refractive index given by Eq. 9.161, and with only a single resonant frequency. Suppose that the absorption of light is given by the optical depth, τ: for incident intensity I_{in}, the outgoing intensity is $I_{out} = I_{in}e^{-\tau}$. (Note: For our purposes, intensity is equal to the magnitude of the Poynting vector). What is the maximum phase shift, $\Delta\phi = \int (n-1)kdz$, produced by refraction, at any frequency, for this material?

3 HW4 3:

Find the dispersion relation for *longitudinal* electric waves in a conductor, with conductivity σ, dielectric constant ϵ, and magnetic permittivity μ. Show that the charge density ρ is nonzero for such a wave, and find how it is related to the longitudinal electric field of the wave $\vec{E}$. Show that the frequency ω is independent of wavevector k. Find the real and imaginary parts of ω. Does the solution admit exponentially growing modes, as well as exponentially declining ones? Explain. Show that the magnetic field for such waves remains zero at all times.

4 Problems from Griffiths

9.12, 9.18, 9.23, 9.24, 9.25