Phys 110C: Problems for HW 7

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1 HWX 1: Poynting Vector for a Standing Wave

In a region of vacuum, a standing electromagnetic wave has the electric field:

$$\vec{E}_{SW}(\vec{r}, t) = E_0 \hat{x} \cos(kz) \exp(-i\omega t). \tag{1}$$

Note that the physical electric field is the real part of the right-hand side of this equation. (Note: Part a of this problem was a midterm problem in 2008).

a)

- Show that the wave can be expressed as the sum of 2 propagating waves.
- Find the magnetic field for the wave.

b)

- Find the energy density in E and B fields.
- Find the Poynting vector for the wave.

(Hint: Be careful!)

2 HWX 2: Magnetic Dipole

Consider a current ring, of radius R and width w. The ring lies flat in the $x-y$ plane, and is centered on the origin. The ring carries a surface current $\vec{K} = K_0 \hat{\phi} \cos(\omega t) = K_0 \hat{\phi} \exp(-i\omega t)$, where in the final expression we use the convention that the real part of the complex number represents the physical quantity. The ring has zero charge. Suppose that the radius of the ring is much smaller than

a) Integrate $\vec{K}$ to find the vector potential $\vec{A}(\vec{r}_F, t_F)$ from the ring at a field point $\vec{r}_F = (r_F \sin\theta_F, 0, r_F \cos\theta_F)$, at large distance $|\vec{r}_F|$. You may use the approximation derived in class (see also Jackson, Sec 9.1, Eq 9.8 and surrounding text):

$$\vec{A}(\vec{r}_F, t_F) \approx \frac{\mu_0}{4\pi} \frac{\exp\left\{i\left(\frac{\omega}{c} r_F - \omega t\right)\right\}}{r_F} \int d^3r' \, \vec{J}(\vec{r}\,') \, \exp\left\{-i\frac{\omega}{c}\vec{r}\,' \cdot \hat{r}_F\right\}. \tag{2}$$

Here, $\hat{r}_F$ is a unit vector that points from the origin toward the field point. This equation is also discussed in Sec 8.7 of Marion & Heald (see Eq 8.100 – note that Marion & Heald use Gaussian units, resulting in no constant in front).

Also argue that $V = 0$.

b) Find the electric and magnetic fields for the vector potential derived in part a. Find the Poynting vector, and the total radiated power P_{rad}.

c) The derivation of the expression for $\vec{A}$ above, Eq. 2, makes the assumption that $d << r_F$, where $\ell = \text{Max}\{|r'|\}$. This allows the approximation $|\vec{r}_F - \vec{r}\,'| \approx |\vec{r}_F|$ in the denominator in the expression for $\vec{A}$ (see, for example Griffiths 11.26, or Jackson 9.3) As Marion points out, it also requires the assumption that r_F is in the "Fraunhofer limit" of $r_F >> \ell^2/(2\lambda)$. Why is this assumption important? Explain.

3 HWX 3: Wave Reflection and Transmission

A plane wave initially travels through vacuum toward $+\hat{z}$. The electric field of the wave is

$$\vec{E}_I(\vec{r}, t) = E_0 \hat{x} e^{i(k_0 z - \omega t)}.$$

Thus, the wavevector is $\vec{k}_I = k_0 \hat{z}$, where $k_0 = \omega/c$.

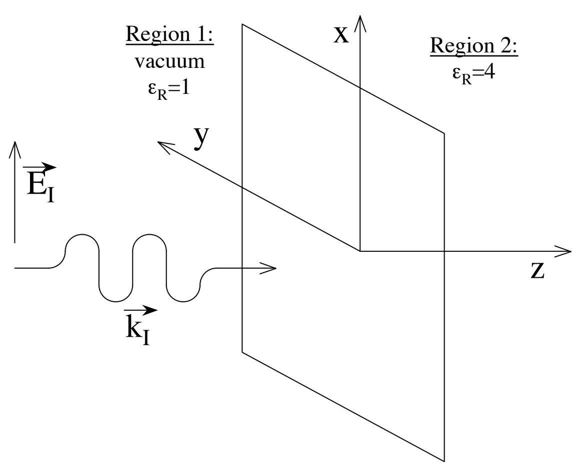
a)

- Find the amplitude of the magnetic field of the wave B_0, and its direction.
- Find the *time-averaged* Poynting vector for the wave, $\vec{S}$.

b) At $z = 0$, the wave is normally incident on the surface of a material with relative dielectric constant $\epsilon_R = 4$. This leads to a reflected wave, traveling toward $-\hat{z}$ and with electric-field amplitude E_{R0}; and a transmitted wave, traveling toward $+\hat{z}$ with electric-field amplitude E_{T0}.

- Find the magnitude of the wavevector in the material, $|\vec{k}_T|$.
- Give the *directions* of
  - the wavevectors $\vec{k}_R$, $\vec{k}_T$,
  - of the E-fields $\vec{E}_R$, $\vec{E}_T$, and
  - of the B-fields $\vec{B}_R$, $\vec{B}_T$

  for the reflected and transmitted waves.
- Write down the condition that relates the amplitudes of E-fields $|\vec{E}_I|$, $|\vec{E}_R|$, and $|\vec{E}_T|$. Write down the condition that relates amplitudes $|\vec{B}_I|$, $|\vec{B}_R|$, and $|\vec{B}_T|$. Which of Maxwell's Equations define these relations?



c) Using your results from part b:

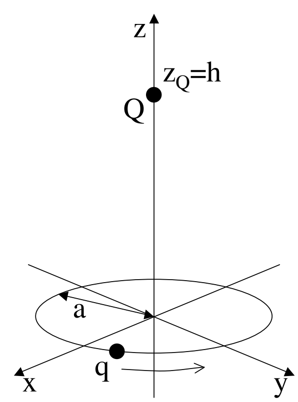
- Find the amplitudes of the electric fields of the reflected and transmitted waves, in terms of E_0, k_0, and $\epsilon_R = 4$.

- Check your result with the fact that $|\vec{E}_I||\vec{B}_I| = |\vec{E}_R||\vec{B}_R| + |\vec{E}_T||\vec{B}_T|$. Why is this the case?

d) Now suppose that the material at $z > 0$ has conductivity σ. Otherwise, the situation is just as it was in parts b and c: the wave is incident normally, and $\epsilon_r = 4$.

- Assume time dependence of the form $e^{-i\omega t}$, and space dependence of the form e^{ikx}, with k complex. Write down a dispersion relation, relating k and ω.
- Assume that the conductivity is poor, in the sense that $\sigma << \epsilon_R \epsilon_0 \omega$. Find an expression for k in terms of ω, σ, ϵ_R, and the constants of electromagnetism. (Possibly useful note: $\sqrt{a+\epsilon} \approx \sqrt{a} + \frac{1}{2}\frac{\epsilon}{\sqrt{a}}$, for $|\epsilon| << a$, even if ϵ is complex.)
- Briefly describe the possible forms of waves inside the conductor.

4 HWX 4: Revolving Charge



a) A charge q revolves in a circle of radius a, in the $x-y$ plane about the z-axis, at angular frequency ω:

$$x(t) = a\cos(\omega t)$$
$$y(t) = a\sin(\omega t).$$

A test charge Q resides at $z_Q = h$ on the z-axis. At time t_Q:

- Find the retarded time t_R at the charge q,

- find the location of q at t_R, and
- find the velocity of q at t_R.

d)

- Find the electric potential $V(h, t_Q)$ at the test charge.
- Find the vector potential $\vec{A}(h, t_Q)$ at the test charge.

5 HWX 5: Plasma Refraction

Recall that in a plasma (such as a highly-conducting metal), the dispersion relation is

$$c^2k^2 = \omega^2 - \omega_p^2$$

where ω_p is the plasma frequency, a constant for the material.

- From the dispersion relation, find the phase velocity v_ϕ for electromagnetic waves in a plasma, as a function of frequency ω.
- Find the group velocity v_G for electromagnetic waves in a plasma, as a function of frequency ω.
- For $\omega > \omega_p$, which velocity, phase or group, is greater than c? Which is less than c? Briefly discuss this result.