
Homework 3 — Solutions 4/21/2004 by Sue Ann Koay

Problem 8.4

(a) As usual we look for eigenfunctions via separation of variables:

$$
\begin{aligned}
\psi(\varpi,\phi) &= R(\varpi)\,Q(\phi) \\
0 &= \left(\nabla_t^2+\gamma^2\right)\psi = \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\psi}{\partial\varpi}\right)+\frac{1}{\varpi^2}\frac{\partial^2\psi}{\partial\phi^2}+\gamma^2\psi \quad , \quad \gamma^2 \equiv \mu\,\epsilon\,\omega^2-k^2 \\
&= R''\,Q+\frac{1}{\varpi}R'\,Q+\frac{1}{\varpi^2}R\,Q''+\gamma^2 R\,Q \\
-\frac{Q''}{Q} &= \varpi^2\frac{R''}{R}+\varpi\frac{R'}{R}+\gamma^2\varpi^2 = \nu^2 = \text{constant}
\end{aligned}
$$

$$
\begin{aligned}
R''+\frac{1}{\varpi}R'+\left(\gamma^2-\frac{\nu^2}{\varpi^2}\right)R &= 0 \implies R = A\,J_\nu(\gamma\,\varpi)+B\,N_\nu(\gamma\,\varpi) \\
Q''+\nu^2 Q &= 0 \implies Q = C\,e^{i\,\nu\,\phi}+D\,e^{-i\,\nu\,\phi}
\end{aligned}
$$

Since the full range $0\le\phi\le 2\pi$ is allowed, single-valuedness in ϕ requires $\nu = m\in\mathbb{Z}$. Finiteness as $\varpi\to 0$ requires $B=0$. The different modes satisfy the boundary conditions 8.36 with surface the cylinder wall $\varpi = R$:

Mode	TM		TE	
Conditions	$\psi\vert_{\varpi=R} = 0 = R\vert_{\varpi=R} = A\,J_m(\gamma\,R)$ $\gamma_{mn} = x_{mn}/R$ $J_m(x_{mn}) = 0 \ , \ n\in\mathbb{Z}^+$		$\left.\frac{\partial\psi}{\partial\varpi}\right\vert_{\varpi=R} = 0$ $\gamma_{mn} = y_{mn}/R$ $J_m'(y_{mn}) = 0 \ , \ n\in\mathbb{Z}^+$	
Cutoff	$\omega_{mn}^{\text{TM}} = \gamma_{mn}/\sqrt{\mu\,\epsilon} = \frac{x_{mn}}{R\sqrt{\mu\,\epsilon}}$		$\omega_{mn}^{\text{TE}} = \frac{y_{mn}}{R\sqrt{\mu\,\epsilon}}$	
Numerical Values	$\omega_{01}^{\text{TM}} \approx 2.405/R\sqrt{\mu\,\epsilon}$		$\omega_{11}^{\text{TE}} \approx 1.841/R\sqrt{\mu\,\epsilon}$	
	$\omega_{11}^{\text{TM}} \approx 1.593\,\omega_{01}^{\text{TM}}$	$\approx 3.831/R\sqrt{\mu\,\epsilon}$	$\omega_{21}^{\text{TE}} \approx 1.659\,\omega_{11}^{\text{TE}}$	$\approx 3.054/R\sqrt{\mu\,\epsilon}$
	$\omega_{21}^{\text{TM}} \approx 2.136\,\omega_{01}^{\text{TM}}$	$\approx 5.137/R\sqrt{\mu\,\epsilon}$	$\omega_{31}^{\text{TE}} \approx 2.282\,\omega_{11}^{\text{TE}}$	$\approx 4.201/R\sqrt{\mu\,\epsilon}$
	$\omega_{12}^{\text{TM}} \approx 2.295\,\omega_{01}^{\text{TM}}$	$\approx 5.519/R\sqrt{\mu\,\epsilon}$	$\omega_{41}^{\text{TE}} \approx 2.888\,\omega_{11}^{\text{TE}}$	$\approx 5.317/R\sqrt{\mu\,\epsilon}$
	$\omega_{31}^{\text{TM}} \approx 2.653\,\omega_{01}^{\text{TM}}$	$\approx 6.380/R\sqrt{\mu\,\epsilon}$	$\omega_{12}^{\text{TE}} \approx 2.896\,\omega_{11}^{\text{TE}}$	$\approx 5.332/R\sqrt{\mu\,\epsilon}$

(b) Expanding in eigenmodes:

$$\begin{aligned}
\psi(\varpi,\phi) &= \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{i\,m\,\phi}\, J_m(\gamma_{mn}\,\varpi) \\
\vec{\nabla}_t\psi(\varpi,\phi) &= \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{i\,m\,\phi}\left[\gamma_{mn}\, J'_m(\gamma_{mn}\,\varpi)\,\hat{\varpi} + \frac{i\,m}{\varpi}\, J_m(\gamma_{mn}\,\varpi)\,\hat{\phi}\right]
\end{aligned}$$

From the table in (a), the lowest mode is $\omega_{11}^{\mathrm{TE}}$, for which the transmitted power 8.51 is

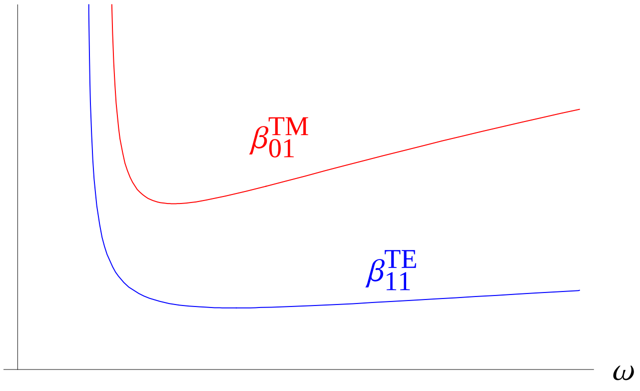
$$\begin{aligned}
P &= \frac{1}{2\sqrt{\mu\,\epsilon}}\left(\frac{\omega}{\omega_{11}^{\mathrm{TE}}}\right)^2\sqrt{1-\left(\frac{\omega_{11}^{\mathrm{TE}}}{\omega}\right)^2}\;\mu\int_0^R d\varpi\int_0^{2\pi}\varpi\, d\phi\,\left|A_{11}\, e^{i\,\phi}\, J_1(\gamma_{11}\,\varpi)\right|^2 \\
&= \pi\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{11}^{\mathrm{TE}}}\right)^2\sqrt{1-\left(\frac{\omega_{11}^{\mathrm{TE}}}{\omega}\right)^2}\;|A_{11}|^2\int_0^R d\varpi\;\varpi\left|J_1\!\left(y_{11}\,\frac{\varpi}{R}\right)\right|^2 \\
&= \pi\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{11}^{\mathrm{TE}}}\right)^2\sqrt{1-\left(\frac{\omega_{11}^{\mathrm{TE}}}{\omega}\right)^2}\;|A_{11}|^2\,\frac{R^2}{2}\,|J_2(y_{11})|^2 \\
-\frac{dP}{dz} &= \frac{1}{2\,\sigma\,\delta}\left(\frac{\omega}{\omega_{11}^{\mathrm{TE}}}\right)^2\int_0^{2\pi} R\, d\phi\left[\frac{1-(\omega_{11}^{\mathrm{TE}}/\omega)^2}{\mu\,\epsilon\,(\omega_{11}^{\mathrm{TE}})^2}\left|\hat{\varpi}\times\vec{\nabla}_t\psi_{11}\right|^2+\left(\frac{\omega_{11}^{\mathrm{TE}}}{\omega}\right)^2|\psi_{11}|^2\right]_{\varpi=R} \\
&= \frac{\pi\,(\omega/\omega_{11}^{\mathrm{TE}})^2}{\sigma\,\delta}\,R\left[\frac{1-(\omega_{11}^{\mathrm{TE}}/\omega)^2}{\mu\,\epsilon\,(\omega_{11}^{\mathrm{TE}})^2}\,|A_{11}|^2\,\frac{|J_1(y_{11})|^2}{R^2}+\left(\frac{\omega_{11}^{\mathrm{TE}}}{\omega}\right)^2|A_{11}|^2\,|J_1(y_{11})|^2\right] \\
&= \frac{\pi\,R}{\sigma\,\delta}\left(\frac{\omega}{\omega_{11}^{\mathrm{TE}}}\right)^2\left[\frac{1-(\omega_{11}^{\mathrm{TE}}/\omega)^2}{R^2\,\mu\,\epsilon\,(\omega_{11}^{\mathrm{TE}})^2}+\left(\frac{\omega_{11}^{\mathrm{TE}}}{\omega}\right)^2\right]|A_{11}|^2\,|J_1(y_{11})|^2
\end{aligned}$$

$$\boxed{\beta_{11}^{\mathrm{TE}} = -\frac{1}{2\,P}\,\frac{dP}{dz} = \sqrt{\frac{\epsilon\,\omega}{2\,\sigma}}\;\frac{1}{R}\left[1-\left(\frac{\omega_{11}^{\mathrm{TE}}}{\omega}\right)^2\right]^{-1/2}\left[\frac{1-(\omega_{11}^{\mathrm{TE}}/\omega)^2}{R^2\,\mu\,\epsilon\,(\omega_{11}^{\mathrm{TE}})^2}+\left(\frac{\omega_{11}^{\mathrm{TE}}}{\omega}\right)^2\right]}$$

Similarly for the second lowest mode $\omega_{01}^{\mathrm{TM}}$:

$$\begin{aligned}
P &= \frac{1}{2\sqrt{\mu\,\epsilon}}\left(\frac{\omega}{\omega_{01}^{\mathrm{TM}}}\right)^2\sqrt{1-\left(\frac{\omega_{01}^{\mathrm{TM}}}{\omega}\right)^2}\;\epsilon\int_0^R d\varpi\int_0^{2\pi}\varpi\, d\phi\,\left|A_{01}\, e^{i\,\phi}\, J_0(\gamma_{01}\,\varpi)\right|^2 \\
&= \pi\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{01}^{\mathrm{TM}}}\right)^2\sqrt{1-\left(\frac{\omega_{01}^{\mathrm{TM}}}{\omega}\right)^2}\;|A_{01}|^2\,\frac{R^2}{2}\,|J_1(x_{01})|^2 \\
-\frac{dP}{dz} &= \frac{1}{2\,\sigma\,\delta}\left(\frac{\omega}{\omega_{01}^{\mathrm{TM}}}\right)^2\int_0^{2\pi} R\, d\phi\;\frac{1}{\mu^2\,(\omega_{01}^{\mathrm{TM}})^2}\left|\frac{\partial\psi}{\partial\varpi}\right|^2_{\varpi=R} \\
&= \frac{\pi\,R}{\sigma\,\delta}\left(\frac{\omega}{\omega_{01}^{\mathrm{TM}}}\right)^2\frac{1}{\mu^2\,(\omega_{01}^{\mathrm{TM}})^2}\,|A_{01}|^2\,\mu\,\epsilon\left(\omega_{01}^{\mathrm{TM}}\right)^2|J'_0(x_{01})|^2 \;\Longleftarrow\; J'_0 = -J_1
\end{aligned}$$

$$\boxed{\beta_{01}^{\text{TM}} = -\frac{1}{2\,P}\,\frac{dP}{dz} = \sqrt{\frac{\epsilon\,\omega}{2\,\sigma}}\,\frac{1}{R}\left[1-\left(\frac{\omega_{01}^{\text{TM}}}{\omega}\right)^2\right]^{-1/2}}$$



Problem 8.6

(a) This has been solved in §8.7 with $d \to L$. Equations 8.81 and 8.83 give the TM and TE resonant frequencies after dividing by $1/\sqrt{\mu\,\epsilon}\,R$ to get them in the units we want. The values of x_{mn} we need are after 3.92 and the values of x'_{mn} are after 8.82. The lowest four modes have been listed after the relevant equation. It's a bit ambiguous here but I'll take "lowest" to mean those with numerically smallest values at $R/L = 0$, since the $p \neq 0$ modes obviously depend on R/L. This seems to be the meaning implied by the statement before 8.84 regarding ω_{111}.

$$\frac{\omega_{mnp}^{\text{TM}}}{1/\sqrt{\mu\,\epsilon}\,R} = \sqrt{x_{mn}^2 + p^2\,\pi^2\left(\frac{R}{L}\right)^2} \qquad \frac{\omega_{mnp}^{\text{TE}}}{1/\sqrt{\mu\,\epsilon}\,R} = \sqrt{{x'_{mn}}^2 + p^2\,\pi^2\left(\frac{R}{L}\right)^2}$$

$$m,\,p \in \mathbb{Z}^{\text{nonnegative}}\,,\quad n \in \mathbb{Z}^+ \qquad n,\,p \in \mathbb{Z}^{\text{nonnegative}}\,,\quad m \in \mathbb{Z}^+$$

$$\frac{\omega_{010}^{\text{TM}}}{1/\sqrt{\mu\,\epsilon}\,R} = 2.405 \qquad \frac{\omega_{111}^{\text{TE}}}{1/\sqrt{\mu\,\epsilon}\,R} \approx 1.841\sqrt{1+2.912\left(\frac{R}{L}\right)^2}$$

$$\frac{\omega_{110}^{\text{TM}}}{1/\sqrt{\mu\,\epsilon}\,R} = 3.832 \qquad \frac{\omega_{211}^{\text{TE}}}{1/\sqrt{\mu\,\epsilon}\,R} = 3.054\sqrt{1+1.058\left(\frac{R}{L}\right)^2}$$

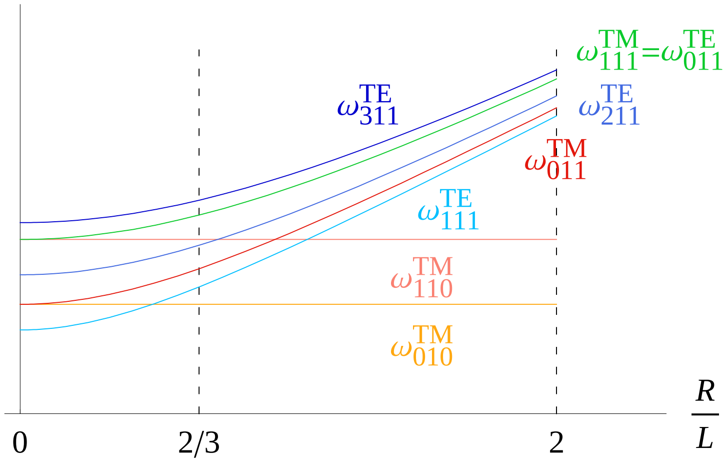
$$\frac{\omega_{011}^{\text{TM}}}{1/\sqrt{\mu\,\epsilon}\,R} \approx 2.405\sqrt{1+1.706\left(\frac{R}{L}\right)^2} \qquad \frac{\omega_{011}^{\text{TE}}}{1/\sqrt{\mu\,\epsilon}\,R} = 3.832\sqrt{1+0.6721\left(\frac{R}{L}\right)^2}$$

$$\frac{\omega_{111}^{\text{TM}}}{1/\sqrt{\mu\,\epsilon}\,R} \approx 3.832\sqrt{1+0.6721\left(\frac{R}{L}\right)^2} \qquad \frac{\omega_{311}^{\text{TE}}}{1/\sqrt{\mu\,\epsilon}\,R} = 4.201\sqrt{1+0.5592\left(\frac{R}{L}\right)^2}$$

To see where lines cross, we solve (for p_1, p_2 not both 0, i.e. not both constant modes where behavior is trivial)

$$\omega_{m_1 n_1 p_1}^{\text{TM}} = \omega_{m_2 n_2 p_2}^{\text{TM}} \implies \left(\frac{R}{L}\right)_{\text{critical}} = \frac{1}{\pi}\sqrt{\frac{x_{m_2 n_2}^2 - x_{m_1 n_1}^2}{p_1^2 - p_2^2}}$$

Supposing without loss of generality that $\omega^{\text{TM}}_{m_1 n_1 p_1}$ was the "lower" mode at $R/L = 0$, so $x^2_{m_2 n_2} - x^2_{m_1 n_1} > 0$, we see that $\omega^{\text{TM}}_{m_2 n_2 p_2}$ may still be the lower mode after $(R/L)_{\text{critical}}$ provided $p_1^2 - p_2^2 > 0$. This is certainly possible, though it doesn't happen in the restricted range of the diagram. The same story goes for the TE modes.



(b) For $R/L = 2/3$ (conveniently marked in the diagram above) the lowest mode is $\omega^{\text{TM}}_{010} = 2.405 / \sqrt{\mu \epsilon}\, R$. Using ψ from 8.80 with $m = 0$, we plug it in to the upper line of 8.92 with $d \to L$ and $p = 0$ (including that extra factor of 2 mentioned in the paragraph after it):

$$\begin{aligned}
\psi(r, \phi) &= E_0\, J_0\!\left(\frac{x_{01}\, r}{R}\right) \\
U &= 2 \times \frac{L}{4}\, \epsilon \int_{\text{cross-section}} da\, |\psi|^2 \;=\; \frac{L}{2}\, \epsilon \int_0^R dr \int_0^{2\pi} r\, d\phi\, |E_0|^2\, J_0^2\!\left(\frac{x_{01}\, r}{R}\right) \\
&= \pi\, L\, \epsilon\, |E_0|^2 \int_0^R dr\; r\, J_0^2\!\left(\frac{x_{01}\, r}{R}\right) \;=\; \pi\, L\, \epsilon\, |E_0|^2\, \frac{R^2}{2}\, J_1^2(x_{01}) \quad \Longleftarrow \quad 3.95
\end{aligned}$$

From 8.82 we read off the $\vec{H}$ field (there is no z component for TM fields), which we plug into 8.93:

$$\begin{aligned}
\vec{H} &= -i \sqrt{\frac{\epsilon}{\mu}}\, E_0\, J_1\!\left(\frac{x_{01}\, r}{R}\right) e^{-i\,\omega\, t}\, \hat{\phi} \\
P_{\text{loss}} &= \frac{1}{2\,\sigma\,\delta} \left(\oint_{\text{circumference}} dl \int_0^L dz\, \left|\hat{r} \times \vec{H}\right|^2_{\text{sides}} + 2 \int_{\text{cross-section}} da\, \left|\hat{r} \times \vec{H}\right|^2_{\text{endcaps}} \right) \\
&= \frac{1}{2\,\sigma\,\delta} \left(\oint_{\text{circumference}} dl \int_0^L dz\, \frac{\epsilon}{\mu}\, |E_0|^2\, J_1^2\!\left(\frac{x_{01}\, R}{R}\right) + 2 \int_{\text{cross-section}} da\, \frac{\epsilon}{\mu}\, |E_0|^2\, J_1^2\!\left(\frac{x_{01}\, r}{R}\right) \right)
\end{aligned}$$

The first integral is trivial. Doing the ϕ integral in the second leaves an r integral which is in my CRC Standard Mathematical Tables and Formulae handbook (you can probably derive this using some variant of the tricks we used in problem 3.11 last quarter, and/or using the recursion relations 3.87 and 3.88):

$$
\begin{aligned}
\int dr\, J_\nu^2(a\,r) &= \frac{r^2}{2}\left[J_\nu^2(a\,r) - J_{\nu-1}(a\,r)\,J_{\nu+1}(a\,r)\right] \\
P_{\text{loss}} &= \frac{1}{2\,\sigma\,\delta}\left(2\,\pi\,R\,L\,\frac{\epsilon}{\mu}\,|E_0|^2\,J_1^2(x_{01}) + 2\,\frac{\epsilon}{\mu}\,2\,\pi\,|E_0|^2\left[\frac{r^2}{2}\,J_1^2\!\left(\frac{x_{01}\,r}{R}\right) - \frac{r^2}{2}\,J_0\!\left(\frac{x_{01}\,r}{R}\right)J_2\!\left(\frac{x_{01}\,r}{R}\right)\right]_0^R\right) \\
&= \frac{\pi\,\epsilon}{\sigma\,\delta\,\mu}\,|E_0|^2\left(R\,L\,J_1^2(x_{01}) + \left[R^2\,J_1^2(x_{01}) - R^2\,J_0(x_{01})\,J_2(x_{01})\right]\right) \\
&= \frac{\pi\,\epsilon}{\sigma\,\delta\,\mu}\,|E_0|^2\,R^2\left(\frac{L}{R} + 1\right)J_1^2(x_{01})
\end{aligned}
$$

Finally we put things together to get Q as in 8.86:

$$
Q = \omega_{010}^{\text{TM}}\,\frac{U}{P_{\text{loss}}} = \frac{\pi\,L\,\epsilon\,|E_0|^2\,\frac{R^2}{2}\,J_1^2(x_{01})\,\omega_{010}^{\text{TM}}}{\frac{\pi\,\epsilon}{\sigma\,\delta\,\mu}\,|E_0|^2\,R^2\left(\frac{L}{R}+1\right)J_1^2(x_{01})} = \boxed{\frac{\sigma\,\delta\,\mu\,L\,\omega_{010}^{\text{TM}}}{2\,(L/R+1)}}
$$

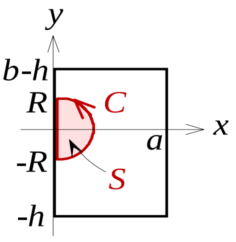
For the numerical values let's assume $\mu \approx \mu_0$ and $\epsilon \approx \epsilon_0$, and room temperature so that the conductivity for copper is $\sigma \approx 5.952\times10^7\,/\,\Omega\cdot m$ as given after 5.165. Inserting δ from 8.8 and ω_{010}^{TM} from above:

$$
\begin{aligned}
\omega_{010}^{\text{TM}} &\approx \frac{x_{01}}{\sqrt{\mu_0\,\epsilon_0}\;R} = \frac{x_{01}\,c}{R} \approx \frac{2.405\,(3.0\times10^8\,m/s)}{2\times10^{-2}\,m} \approx 3.608\times10^{10}\,s^{-1} \\
\delta &\approx \left(\frac{2}{\mu_0\,\omega_{010}^{\text{TM}}\,\sigma}\right)^{1/2} \approx \left(\frac{2}{(4\,\pi\times10^{-7}\,N/A^2)\,(36.08\,s^{-1})\,(5.952\times10^7\,/\,\Omega\cdot m)}\right)^{1/2} \approx 8.609\times10^{-7}\,m \\
Q &\approx \frac{\sigma\,\delta\,\mu_0\,L\,\omega_{010}^{\text{TM}}}{2\,(L/R+1)} \approx \frac{\left(\frac{5.952\times10^7}{\Omega\cdot m}\right)(8.609\times10^{-7}\,m)\left(4\,\pi\times10^{-7}\,\frac{N}{A^2}\right)(3\times10^{-2}\,m)\,(3.608\times10^{10}\,s^{-1})}{2\,(3/2+1)} \\
&\approx \boxed{1.394\times10^4}
\end{aligned}
$$

Problem 8.20

(a) Arbitrarily we choose the coordinate system as illustrated. From 8.146:

$$
\begin{aligned}
\vec{J} &= I_0\,\delta(z)\,\delta(\varpi - R)\,\hat{\phi} \quad (I_0 > 0 \text{ for counterclockwise current}) \\
A_\lambda^{\pm} &= -\frac{Z_\lambda}{2}\int_V d^3x\;\vec{J}\cdot\vec{E}_\lambda^{\mp} \\
&= -\frac{Z_\lambda}{2}\int dz\int d\varpi\int_{-\pi/2}^{\pi/2}\varpi\,d\phi\;I_0\,\delta(z)\,\delta(\varpi - R)\,\hat{\phi}\cdot\vec{E}_\lambda^{\mp} \\
&= -\frac{Z_\lambda\,I_0\,R}{2}\int_{-\pi/2}^{\pi/2} d\phi\;\hat{\phi}\cdot\vec{E}_\lambda^{\mp}\Big|_{\varpi=R,z=0}
\end{aligned}
$$



From inspection of 8.135 we see that $\vec{E}_\lambda\cdot\hat{y}\big|_{x=0} = 0$, so we can gratuitously add the line integral along the y-axis from R to $-R$, converting the line integral above to one over the closed semicircular loop C:

$$A_\lambda^\pm \;\propto\; \oint_C \vec{E}_\lambda^\mp \cdot d\vec{l} \quad , \quad d\vec{l} \text{ the directed line element along } C$$

The TM modes have transverse fields that satisfy $\vec{E}_\lambda^\mp \propto \vec{\nabla}_t \psi$ (we'll ignore all the irrelevant constants, labels, etc.). Converting the line integral into a surface integral via Stokes' theorem:

$$A_\lambda^\pm \;\propto\; \int_S da \left[\vec{\nabla} \times \left(\vec{\nabla}_t \psi\right)\right] \cdot \hat{z} \;=\; \int_S da \left[\vec{\nabla} \times \left(\vec{\nabla} \psi - \frac{\partial \psi}{\partial z}\,\hat{z}\right)\right] \cdot \hat{z}$$

Using $\vec{\nabla} \times \vec{\nabla} \psi = 0$ gets rid of the first term. The remaining $\vec{\nabla} \times (\partial_z \psi\, \hat{z})$ has no z component, and so vanishes as well when dotted into $\hat{z}$. Thus we have shown that $A_\lambda^\pm = 0$. Physically we should have expected this because the constant-current half-loop creates (from magnetostatics) magnetic fields in the z direction, which is not compatible with the TM condition that $B_z = 0$ everywhere.

(b) From 8.44 we see that the lowest mode is ω_{01} if $a < b$, ω_{10} if $a > b$. The calculations are similar so let's just do one, say ω_{10} since it looks like $a > b$ from the picture. 8.136 gives the TE waves, with extra factor if $1 \big/ \sqrt{2}$ as per discussion in the paragraph after 8.136, $\gamma_{10} = \pi / a$ from 8.43, and $y \to y + h$ due to the shifted coordinate system (not that it matters). Again the z and ϖ integrals are trivial:

$$\vec{E}_{10} \;=\; \frac{2\pi}{(\pi / a)\, a \sqrt{2\,a\,b}} \sin\!\left(\frac{\pi\,x}{a}\right) \hat{y}$$

$$A_{10}^\pm \;=\; -\frac{Z_{10}}{2} \int_V d^3x\; I_0\, \delta(z)\, \delta(\varpi - R)\, \hat{\phi} \cdot \vec{E}_\lambda^\mp \;=\; -\frac{Z_{10}\, I_0}{\sqrt{2\,a\,b}} \int_{-\pi/2}^{\pi/2} R\, d\phi\, \sin\!\left(\frac{\pi\,R \cos\phi}{a}\right) \cos\phi$$

The expression is manifestly h-independent. The integral can be simplified by first halving the range since the integrand is even under $\phi \to -\phi$, then a change of variables $\cos\phi \equiv u$, so $du = -\sin\phi\, d\phi = -\sqrt{1 - u^2}\; d\phi$:

$$A_{10}^\pm \;=\; +\frac{Z_{10}\, I_0\, R}{\sqrt{2\,a\,b}}\, 2 \int_1^0 du\; \frac{u \sin(\pi\,R\,u / a)}{\sqrt{1 - u^2}}$$

The integral is in Gradshteyn and Ryzhik 3.753(5), which gives $\int_0^1 dx\; x \sin(a\,x)\, (1 - x^2)^{-1/2} = \frac{\pi}{2}\, J_1(a)$:

$$\boxed{A_{10}^\pm \;=\; -\frac{\pi\, Z_{10}\, I_0\, R}{\sqrt{2\,a\,b}}\, J_1\!\left(\frac{\pi\,R}{a}\right) \quad , \quad Z_{10} \;=\; \frac{\mu\, \omega_{10}}{k_{10}}}$$

(c) From 8.133, expanding $J_1(x) \approx (x/2) / \Gamma(2) = x/2$ from 3.89 for $R \ll a, b$:

$$\langle P \rangle \;=\; \frac{1}{2} \int da\, \left(\vec{E}_{10} \times \vec{H}_{10}\right) \cdot \hat{z} \;=\; \frac{1}{2\,Z_{10}}\, |A_{10}^\pm|^2 \;=\; \frac{\pi^2\, Z_{10}\, I_0^2\, R^2}{4\,a\,b}\, J_1^2\!\left(\frac{\pi\,R}{a}\right)$$

$$\boxed{\langle P \rangle \;\approx\; \frac{\pi^2\, Z_{10}\, I_0^2\, R^2}{4\,a\,b} \left(\frac{\pi\,R}{2\,a}\right)^2 \;=\; \frac{I_0^2}{16}\, Z_{10}\, \frac{a}{b} \left(\frac{\pi\,R}{a}\right)^4}$$