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# *Phys 210b Homework 7 — Solutions 5/28/2004 by Sue Ann Koay*

## *Problem 11.30*

Let's put primes on quantities in the rest frame $K'$ moving with constant velocity $\vec{v}$ relative to the lab frame $K$. Thus the (unprimed) quantitites in the $K$-frame are given by reverse-boosting by $-\vec{\beta} = -\vec{v}/c$. Since $G_{\mu\nu}$ and $F_{\mu\nu}$ transform the same way we may use by analogy (11.149) with $\vec{E} \to \vec{D}$ and $\vec{B} \to \vec{H}$ everywhere:

$$\begin{aligned}
\vec{E} &= \gamma\left(\vec{E}' - \vec{\beta}\times\vec{B}'\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}'\right) \quad ; \quad \vec{B}' = \gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right) \\
\vec{D} &= \gamma\left(\vec{D}' - \vec{\beta}\times\vec{H}'\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{D}'\right) = \gamma\left(\epsilon\vec{E}' - \frac{1}{\mu}\,\vec{\beta}\times\vec{B}'\right) - \frac{\gamma^2}{\gamma+1}\,\epsilon\vec{\beta}\left(\vec{\beta}\cdot\vec{E}'\right) \\
&= \epsilon\Big[\underbrace{\gamma\left(\vec{E}' - \vec{\beta}\times\vec{B}'\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}'\right)}_{\vec{E}} + \gamma\,\vec{\beta}\times\vec{B}' - \frac{\gamma}{\mu\,\epsilon}\,\vec{\beta}\times\vec{B}'\Big] \\
&= \epsilon\vec{E} + \left(\epsilon - \frac{1}{\mu}\right)\gamma\,\vec{\beta}\times\left[\gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)\right]
\end{aligned}$$

$\vec{\beta}\times\vec{\beta} = 0$ so the last term vanishes. The second formula (front cover) says $\vec{\beta}\times\left(\vec{\beta}\times\vec{E}\right) = \left(\vec{\beta}\cdot\vec{E}\right)\vec{\beta} - \left(\vec{\beta}\cdot\vec{\beta}\right)\vec{E}$. Rewriting $\vec{E} = E_\perp\,\hat{\beta}_\perp + E_\parallel\,\hat{\beta}$ where $\hat{\beta}_\perp$ and $\hat{\beta}$ are unit vectors perpendicular and parallel to $\vec{\beta}$ (and thus $\vec{v}$), this is $\vec{\beta}\times\left(\vec{\beta}\times\vec{E}\right) = E_\parallel\,\beta^2\,\hat{\beta} - \beta^2\left(E_\perp\,\hat{\beta}_\perp + E_\parallel\,\hat{\beta}\right) = -\beta^2\,\vec{E}_\perp$, giving the desired

$$\boxed{\vec{D} = \epsilon\vec{E} + \gamma^2\left(\epsilon - \frac{1}{\mu}\right)\left(\beta^2\,\vec{E}_\perp + \vec{\beta}\times\vec{B}\right)}$$

The calculation for $\vec{H}$ is essentially the same, using $\vec{\beta}\times\left(\vec{\beta}\times\vec{B}\right) = -\beta^2\,\vec{B}_\perp$:

$$\begin{aligned}
\vec{B} &= \gamma\left(\vec{B}' + \vec{\beta}\times\vec{E}'\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{B}'\right) \quad ; \quad \vec{E}' = \gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right) \\
\vec{H} &= \gamma\left(\vec{H}' + \vec{\beta}\times\vec{D}'\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{H}'\right) = \gamma\left(\frac{1}{\mu}\,\vec{B}' + \epsilon\,\vec{\beta}\times\vec{E}'\right) - \frac{\gamma^2}{\gamma+1}\,\frac{1}{\mu}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{H}'\right) \\
&= \frac{1}{\mu}\left[\gamma\left(\vec{B}' + \vec{\beta}\times\vec{E}'\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{H}'\right) - \gamma\,\vec{\beta}\times\vec{E}' + \gamma\,\mu\,\epsilon\,\vec{\beta}\times\vec{E}'\right] \\
&= \frac{1}{\mu}\,\vec{B} + \gamma\left(\epsilon - \frac{1}{\mu}\right)\vec{\beta}\times\left[\gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right)\right] = \boxed{\frac{1}{\mu}\,\vec{B} + \gamma^2\left(\epsilon - \frac{1}{\mu}\right)\left(-\beta^2\,\vec{B}_\perp + \vec{\beta}\times\vec{E}\right)}
\end{aligned}$$

## *Problem 13.1*

(a) The paragraphs above equations (13.2) and (13.3) gives the 4-momentum transfer squared for elastic scattering $Q^2 = -(p - p')^2 = 4\,p^2\sin^2(\theta/2) = 2\,m\,T\,(b)$. Using the standard identity $\sin^2 x = 1/(1 + \cot^2 x)$:

$$b \;=\; \frac{z\,e^2}{p\,v}\cot\!\left(\frac{\theta}{2}\right) \;\Longrightarrow\; \cot\!\left(\frac{\theta}{2}\right) \;=\; \frac{b\,p\,v}{z\,e^2} \;=\; \frac{b}{b_{\min}^{(c)}} \quad , \quad b_{\min}^{(c)} \;\equiv\; \frac{z\,e^2}{p\,v}$$

$$Q^2 \;=\; 4\,p^2\sin^2\!\left(\frac{\theta}{2}\right) \;=\; \frac{4\,p^2}{1+\cot^2(\theta/2)} \;=\; \boxed{\frac{4\,p^2}{1+\left(b\big/b_{\min}^{(c)}\right)^2}}$$

$$T(b) \;=\; \frac{Q^2}{2\,m} \;=\; \frac{1}{2\,m}\,\frac{4\,p^2\left(b_{\min}^{(c)}\right)^2}{\left(b_{\min}^{(c)}\right)^2+b^2} \;=\; \boxed{\frac{2\,z^2\,e^4}{m\,v^2}\,\frac{1}{b^2+\left(b_{\min}^{(c)}\right)^2}}$$

(b) We treat the light particle off which the heavy particle of charge $z\,e$ scatters as approximately stationary. The fields seen by the light particle due to the heavy one is given by equation (11.152) with $q = z\,e$. The change in momentum (a.k.a. impulse) delivered to the light particle is due to the electric force $\vec{F} = e\,\vec{E}$. In particular we are interested in the transverse component $F_t = e\,E_2$ which will give us the transverse impulse:

$$\Delta p \;=\; \int_{-\infty}^{\infty} dt\;F_t \;=\; \int_{-\infty}^{\infty} dt\;\frac{\gamma\,z\,e^2\,b}{(b^2+\gamma^2\,v^2\,t^2)^{3/2}} \;=\; \gamma\,z\,e^2\,b\left[\frac{4\,\gamma^2\,v^2\,t}{4\,b^2\,\gamma^2\,v^2\,\sqrt{b^2+\gamma^2\,v^2\,t^2}}\right]_{-\infty}^{\infty}$$

$$=\; 2\,\gamma\,z\,e^2\,b\,\lim_{t\to\infty}\frac{1}{b^2\,\sqrt{b^2/t^2+\gamma^2\,v^2}} \;=\; \boxed{\frac{2\,z\,e^2}{b\,v}}$$

The integral was in the CRC Standard Mathematical Tables and Formulae.

$$T \;\approx\; \frac{\Delta p^2}{2\,m} \;=\; \frac{1}{2\,m}\left(\frac{2\,z\,e^2}{b\,v}\right)^2 \;=\; \boxed{\frac{2\,z^2\,e^4}{m\,v^2}\;\frac{1}{b^2}}$$

This is the special case $b^2 \gg \left(b_{\min}^{(c)}\right)^2$ of part (a). Given the form $b \propto \cot(\theta/2)$ this corresponds to small $\theta$, exactly as claimed in the problem.

## *Problem 13.2*

Let $\vec{x}(t)$ be the position vector of the particle of charge $e$ and mass $m$. With the spring force, damping, and electric force acting (we ignore the magnetic force) we have the equation of motion

$$\ddot{\vec{x}}(t) + \omega_0^2\,\vec{x}(t) + \Gamma\,\dot{\vec{x}}(t) + \frac{e}{m}\,\vec{E}(\vec{x},\,t) \;=\; 0$$

We make the approximation that $\vec{E}(\vec{x},\,t) \approx \vec{E}\big(\vec{0},\,t\big)$ in the region of interest. Then Fourier-transforming the differential equation just converts time-derivatives to multiplication by $-i\,\omega$:

$$\begin{aligned}
0 &\approx (-i\,\omega)^2\,\tilde{\vec{x}}(\omega) + \omega_0^2\,\tilde{\vec{x}}(\omega) - i\,\omega\,\Gamma\,\tilde{\vec{x}}(\omega) + \frac{e}{m}\,\tilde{\vec{E}}(\omega) \;=\; \left(\omega_0^2 - \omega^2 - i\,\omega\,\Gamma\right)\tilde{\vec{x}}(\omega) + \frac{e}{m}\,\tilde{\vec{E}}(\omega) \\
\tilde{\vec{x}}(\omega) &\approx -\frac{e}{m}\,\frac{\tilde{\vec{E}}(\omega)}{\omega_0^2 - \omega^2 - i\,\omega\,\Gamma} \\
\vec{E}(t) &\equiv \vec{E}\left(\vec{0},\,t\right) \;=\; \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} d\omega\;\tilde{\vec{E}}(\omega)\,e^{-i\,\omega\,t} \quad,\quad \tilde{\vec{E}}(\omega) \;=\; \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} dt\;\vec{E}\left(\vec{0},\,t\right)e^{i\,\omega\,t}
\end{aligned}$$

Note that this wouldn't have worked out so simply if we had the $\vec{x}$-dependent quantity $\vec{E}(\vec{x},\,t)$. The total energy transfer is given by the usual

$$\begin{aligned}
\Delta E &= \int \vec{F}\cdot d\vec{l} \;=\; -\int_{-\infty}^{\infty} dt\;e\,\vec{E}(t)\cdot\dot{\vec{x}}(t) \\
&= -\int_{-\infty}^{\infty} dt\;e\left[\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} d\omega\;\tilde{\vec{E}}(\omega)\,e^{-i\,\omega\,t}\right]\left[\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} d\omega'\;(-i\,\omega')\,\tilde{\vec{x}}(\omega')\,e^{-i\,\omega'\,t}\right] \\
&= \frac{i\,e}{2\,\pi}\int_{-\infty}^{\infty} d\omega\int_{-\infty}^{\infty} d\omega'\;\omega'\,\tilde{\vec{x}}(\omega')\,\tilde{\vec{E}}(\omega)\underbrace{\int_{-\infty}^{\infty} dt\;e^{-i\,(\omega+\omega')\,t}}_{2\,\pi\,\delta(\omega+\omega')} \;=\; -i\,e\int_{-\infty}^{\infty} d\omega\;\omega\,\tilde{\vec{x}}(-\omega)\,\tilde{\vec{E}}\,(\omega) \\
&= -i\,e\int_{-\infty}^{\infty} d\omega\;\omega\,\tilde{\vec{x}}^{*}(\omega)\,\tilde{\vec{E}}(\omega)
\end{aligned}$$

In the last step we've used the fact that since $\vec{x}(t)$ must be real, $\tilde{\vec{x}}(-\omega) = \tilde{\vec{x}}^{*}(\omega)$. Plugging in $\tilde{\vec{x}}(\omega)$:

$$\Delta E \;=\; \frac{i\,e^2}{m}\int_{-\infty}^{\infty} d\omega\;\omega\;\frac{\tilde{\vec{E}}^{*}(\omega)\,\tilde{\vec{E}}(\omega)}{\omega_0^2 - \omega^2 + i\,\omega\,\Gamma} \;=\; \frac{i\,e^2}{m}\int_{-\infty}^{\infty} d\omega\;\frac{\omega}{\omega_0^2 - \omega^2}\left|\tilde{\vec{E}}(\omega)\right|^2 + O(\Gamma)$$

We can do the above integral by contour integration (it is actually quite easy to do the full case with $\Gamma$-dependence, but I'll leave that for you to play with). $\tilde{\vec{E}}(\omega)$ is an analytic function of $\omega$ (being a Fourier transform), so there are only simple poles from the denominator at $\omega = \omega_{\pm} \equiv \pm\omega_0$. Extending the integral to a closed semi-circular contour in the upper-half complex-$\omega$ plane, where $C_R$ is a semi-circular arc of radius $R \to \infty$, we need to verify that the integral on the arc vanishes. Using a result from complex analysis, $\left|\int_C f(z)\,dz\right| \leq \max_C[f]\,L_C$ where $\max_C[f]$ is the maximum the function $f$ takes on the contour $C$ and $L_C$ is the length of the contour:

$$\begin{aligned}
\left|\int_{C_R} ds\;\frac{\omega}{\omega_0^2 - \omega^2}\left|\tilde{\vec{E}}(\omega)\right|^2\right| &\leq \max_{C_R}\left[\frac{\omega}{\omega_0^2 - \omega^2}\left|\tilde{\vec{E}}(\omega)\right|^2\right]\pi\,R \;=\; \max_{C_R}\left[\frac{R\,e^{i\,\theta}}{\omega_0^2 - R^2\,e^{2\,i\,\theta}}\left|\tilde{\vec{E}}\left(R\,e^{i\,\theta}\right)\right|^2\right]\pi\,R \\
&\leq \max_{C_R}\left[\frac{\pi\,R^2\,e^{i\,\theta}}{\omega_0^2 - R^2\,e^{2\,i\,\theta}}\int_{-\infty}^{\infty} dt\;dt'\;\vec{E}(t)\,\vec{E}(t')\,\exp\left(i\,R\,e^{i\,\theta}\,t - i\,R\,e^{-i\,\theta}\,t'\right)\right]
\end{aligned}$$

having made use that $\vec{E}(t)$ is real. We can rewrite $i\,R\,e^{i\,\theta}\,t - i\,R\,e^{-i\,\theta}\,t' = i\,R\,(t-t')\cos\theta - R\,(t+t')\sin\theta$. As $R \to \infty$ the first term results in a rapidly oscillating phase that kills the $t$ (or $t'$) integral [$\vec{E}(t)$ is real and so cannot contain a cancelling phase]. Thus $\lim_{R\to\infty}\int_{C_R} \to 0$ as required, and we're left with

$$
\begin{aligned}
\Delta E &= \frac{i\, e^2}{m}\, \pi\, i \operatorname*{res}_{\omega\to\omega_\pm} \left[ \frac{\omega}{\omega_0^2-\omega^2} \left|\tilde{\vec{E}}(\omega)\right|^2 \right] + \mathcal{O}\,(\Gamma) = \frac{\pi\, e^2}{m} \lim_{\omega\to\omega_\pm} \left[ (\omega-\omega_\pm)\, \frac{\omega \left|\tilde{\vec{E}}(\omega)\right|^2}{(\omega-\omega_+)\,(\omega-\omega_-)} \right] + \mathcal{O}(\Gamma) \\
&= \frac{\pi\, e^2}{m} \left( \frac{\omega_+ \left|\tilde{\vec{E}}(\omega_+)\right|^2}{\omega_+-\omega_-} + \frac{\omega_- \left|\tilde{\vec{E}}(\omega_-)\right|^2}{\omega_--\omega_+} \right) + \mathcal{O}(\Gamma) = \frac{\pi\, e^2}{m}\, \frac{1}{2} \left( \left|\tilde{\vec{E}}(\omega_0)\right|^2 + \left|\tilde{\vec{E}}(-\omega_0)\right|^2 \right) + \mathcal{O}(\Gamma) \\
&= \boxed{\frac{\pi\, e^2}{m} \left|\tilde{\vec{E}}(\omega_0)\right|^2 + \mathcal{O}(\Gamma)}
\end{aligned}
$$

In the first step we've used the form appropriate for poles on the real axis, i.e. half the usual weight for residues. In the last step the reality of $\vec{E}(t)$ implies $\tilde{\vec{E}}(-\omega) = \tilde{\vec{E}}^{*}(\omega)$, so $\left|\tilde{\vec{E}}(-\omega_0)\right|^2 = \left|\tilde{\vec{E}}^{*}(\omega_0)\right|^2 = \left|\tilde{\vec{E}}(\omega_0)\right|^2$.