Scintillations and Interstellar Lévy Flights

Stanislav Boldyrev[1] and Carl R. Gwinn[2]
[1]*Department of Astronomy and Astrophysics, University of Chicago, Chicago, IL 60637*, boldyrev@uchicago.edu
[2]*Department of Physics, University of California, Santa Barbara, CA 93106*, cgwinn@physics.ucsb.edu

Abstract

We present the new model for interstellar scintillations, which assumes that the electron density fluctuations causing the scintillations obey Lévy statistics rather than Gaussian statistics, as was thought previously. Our model explains the observed anomalous scaling of the signal time-width with respect to the pulsar distance, which remained a puzzle for more than 30 years.

1. Introduction. The time-shape of a radio signal arriving from a distant pulsar is distorted due to random refraction caused by electron-density fluctuations in the interstellar medium (ISM). If the emitted pulse is very narrow in time, with the intensity $I(t) \propto \delta(t)$, the observed signal is broad and asymmetric, with a sharp rise and a slow decay. Observations show that broadened shapes of the pulses are similar for different pulsars (after proper rescaling), suggesting that the density fluctuations statistics along different lines of sight are to some extent universal. It is natural to assume that such universality is the signature of the underlying turbulence of the interstellar medium.

The pulse shape, and its scaling with the signal frequency and with the pulsar distance can be evaluated using the geometric optics approach. This is a reasonable approximation since the signal frequency is much larger than the electron plasma frequency of the diffracting medium. For a simple estimate assume that the pulsar distance is $d \sim 10\,\mathrm{kpc}$, the typical electron density is $n \sim 0.03\,\mathrm{cm}^{-3}$, and the observational wave frequency is $\nu \sim 500\,\mathrm{MHz}$. Then the plasma electron frequency, $\omega_{pe} = (4\pi N e^2/m_e)^{1/2} \approx 10^4\,\mathrm{sec}^{-1}$, is much smaller than the wave frequency, and density fluctuations change the wave phase only slightly.

The observed signal is broad since the rays that are strongly deflected travel longer paths and, therefore, arrive at the observer at later times. To estimate the time delay consider the propagating ray that is refracted (scattered) by small prisms of density inhomogeneities [1, 2, 3, 4]. At each scatter event occurring at a mean-free path l, the propagation angle changes by a small amount, $\Delta\theta \sim \lambda^2 r_0 \Delta N$ [see below], where λ is the wave length, $r_0 = e^2/m_e c^2$ is the classical radius of electron, and ΔN is the density difference at the characteristic separation l. The standard models assume that $\Delta\theta$ is random

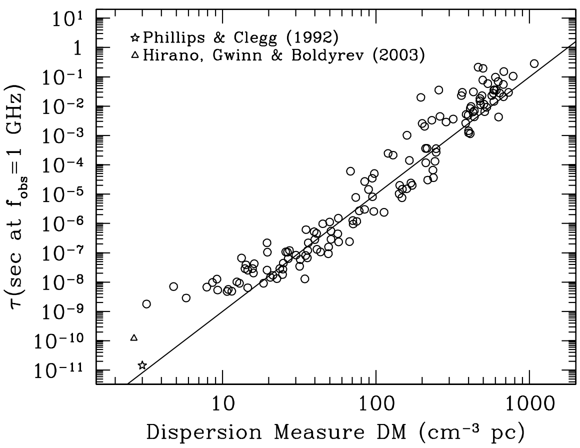

Figure 1: Pulse temporal broadening as the function of the dispersion measure, $\mathrm{DM} = \int_0^d n(z)\,\mathrm{d}z$, which is a measure of the distance to the pulsar [13]. Except as noted, the data were taken from [14]. The solid line has the slope of 4.

and Gaussian, and find that the path direction deviates from a straight line by $\theta \sim (\lambda^4 r_0^2 d/l)^{1/2}$, and the path length deviates from the distance d by $\Delta d \sim d\theta^2 \sim \lambda^4 r_0^2 d^2/l$. The broadening time can be estimated as $\tau_d \sim \Delta d/c$, which gives the standard scaling $\tau_d \propto \lambda^4 d^2$ [1, 2, 6, 7, 8].

Observations show that the signal width, τ_d, estimated at the half-amplitude level, scales with the wave length according to the obtained formula, $\tau_d \propto \lambda^4$, while the scaling with the distance is close to $\tau_d \propto d^4$, which contradicts the analytical prediction, as is seen in Fig. 2 [10]. This paradox was first discussed by Sutton [5], and although the theory of scintillations has been developed for more than 30 years, the contradiction has resisted analytical understanding [6].

The resolution to this paradox was recently proposed in [3]. It was suggested that the anomalous scaling with the distance is the evidence of *non-Gaussian* density fluctuations in the ISM. It was assumed that the probability distribution of density gradients has a *power-law* decay, and its second moment is divergent. Such probability distributions are common in theories of turbulence, which is consistent with the argument that the density statistics are governed by turbulent motions in the ISM [11, 12]. The sum of many angular deviations caused by such fluctuations does not have Gaussian distribution, instead, the limiting distribution is of the Lévy type, and the ray angle performs a Lévy flight instead of a conventional random walk.

2. Rays in a random medium. In the limit of geometric optics, the wave packet trajectory $\mathbf{r}(t)$ obeys the following Hamiltonian system of equations,

$$\begin{aligned} \dot{\mathbf{r}} &= \partial\omega(k,r)/\partial\mathbf{k}, \\ \dot{\mathbf{k}} &= -\partial\omega(k,r)/\partial\mathbf{r}. \end{aligned} \qquad (1)$$

In this formulas, $\omega(k,r)$ plays the role of the Hamiltonian, $\omega^2(k,r) = \omega_{pe}^2(r) + k^2c^2$, where ω_{pe} is the electron plasma frequency introduced above, and k is the characteristic wave vector of the packet. Differentiating the first equation with respect to t and using the second equation in the system (1) we obtain

$$\ddot{\mathbf{r}} = -2\pi c^2\lambda^2 r_0 \partial N(r)/\partial \mathbf{r}, \tag{2}$$

where $r_0 = e^2/(m_e c^2)$ is a classical radius of electron. Therefore, the properties of the ray trajectory depend on the distribution of electron density gradients $\partial N(r)/\partial \mathbf{r}$. The standard approach assumes that these gradients have Gaussian distribution; in the present paper we do not make such an assumption.

Taking into account that the ray propagates at small angles to the line of sight (z axis), we are interested in the ray displacement in the perpendicular, $\mathbf{x}$, direction, and instead of time we use the variable $z = ct$. The ray trajectory can thus be viewed as the function $\mathbf{x}(z)$, and the ray angle is $\boldsymbol{\theta} = d\mathbf{x}/dz$.

In these variables, equation (2) reduces to

$$d\boldsymbol{\theta}/dz = -2\pi\lambda^2 r_0 \partial N(\mathbf{x}, z)/\partial \mathbf{x}. \tag{3}$$

For quantitative estimates we may assume that the total scattering is a sum of elementary scatter events, each occurring on the scale $\Delta z = l \ll d$. At each scattering, the angle deflection is obtained by integrating Eq. (3) over Δz:

$$\Delta\boldsymbol{\theta} = -\pi\lambda^2 r_0 \int^{\Delta z} \mathrm{d}z\boldsymbol{\sigma}(z), \tag{4}$$

where $\boldsymbol{\sigma}(z) = [\partial N(\mathbf{x}, z)/\partial \mathbf{x}]_{\mathbf{x}=\mathbf{x}(z)}$. For the Gaussian $\boldsymbol{\sigma}$, we thus obtain that $\Delta\boldsymbol{\theta}$ are increments of a Gaussian random walk, and therefore $\boldsymbol{\theta}^2 \propto \lambda^4 d$, which reproduces the standard result $\tau_d \propto \lambda^4 d^2$ mentioned in the introduction. This result contradicts the observational scaling $\tau_d \propto \lambda^4 d^4$.

In [3] it was proposed that the anomalous observational d-scaling of the pulse time broadening is the evidence of non-Gaussian statistics of $\Delta\boldsymbol{\theta}$. In other words, this is the evidence of strongly intermittent distribution of electron density gradients. We suggested that such a distribution can arise as the result of ISM turbulence, which also explains the universality of pulse scalings, – such a universality reflect the universality of the turbulence.

As an illustration, let us demonstrate how the intermittent statistics of angle deflections arises in the density field possessing randomly oriented shock discontinuities. Assume that the propagating ray intersects a shock at the angle $\boldsymbol{\varphi}$ to the shock normal, and that the density jump on the shock is ΔN. Then the integral in formula (4) can be easily done, and the small angle deflection is $\Delta\boldsymbol{\theta} = -\pi\lambda^2 r_0 \Delta N \tan(\varphi)\hat{\boldsymbol{\varphi}}$. The same formula can of course be derived from the refraction equation of geometric optics.

We simplify the problem even more, and assume that ΔN is the same for all shocks, and the variables $\mathbf{x}$, $\boldsymbol{\theta}$, and $\boldsymbol{\varphi}$ are one-dimensional. Then the distribution of $\Delta\theta$ is given by the distribution of $y = \tan(\varphi)$ where φ is uniformly distributed in the interval $[-\pi/2, \pi/2]$. It is easy to check that this distribution has the density $P(y) = (1/\pi)/(1 + y^2)$, and is known as the Cauchy distribution. This illustration provides us with an important result. The distribution of deflection angles has a power-law decay, and its second moment is divergent.

Therefore, the sum of many such angle deflections is *not* a Gaussian random walk (the central limit theorem does not hold). Instead, the distribution of such a sum converges to the so-called Lévy distribution that we are discussing in the next section. To conclude this section we note a curious fact that the obtained distribution $P(y)$ is *itself* a Lévy distribution, with $\beta = 1$ (see the definition below).

3. Scintillations as Lévy flights. The Fourier transform (the characteristic function) of a symmetric Lévy distribution $P_\beta(\Delta\theta)$ has a simple form,

$$F(\mu) = \int_{-\infty}^{\infty} \mathrm{d}\Delta\theta \, P_\beta(\Delta\theta) \exp(i\mu\Delta\theta) = \exp\left(-C|\mu|^\beta\right), \tag{5}$$

where $0 < \beta < 2$, and C is some positive constant. Eq. (5) can be taken as the definition of a symmetric Lévy distribution. The sum of n Lévy distributed variables scales as $\sum^n \Delta\theta \sim n^{1/\beta}$, which becomes diffusion in the Gaussian limit $\beta = 2$. For $\beta < 2$, the probability distribution function has algebraic tails, $P_\beta(\Delta\theta) \sim |\Delta\theta|^{-1-\beta}$ for $|\Delta\theta| \to \infty$, and its second moment is divergent. Random walk whose increments are Lévy distributed are called Lévy flights, and they often replace Brownian motion in turbulent system [15]. We thus assume that the angle deflections (corresponding to random density-gradient fluctuations) are Lévy distributed, and are short-scale correlated in z.

For a ray propagating from a point source to a point observer, the time delay is given by

$$\tau = \frac{r_0^2 l^3 \lambda^4}{8\pi^2 c} \sum_{m=1}^{n} \left[\frac{1}{m} \sum_{s=1}^{m} s\sigma_s \right]^2. \tag{6}$$

In this formula, σ_s are density gradients of individual scattering segments of length $\Delta z = l$, which the ray encounters on its way. The total number of such scattering events is $n = d/l$. We will not present the derivation of formula (6) here, it is quite elementary and can be found in [3]. The intensity of the arriving signal has the form $I_\lambda(z,t) = \langle \delta\left(t - d/c - \tau\right) \rangle$, where the averaging is performed over an ensemble of statistically independent rays or equivalently over different realizations of σ's (derivation of this formula can also be found in [3]). This averaging can be done numerically, if we assume that density gradients σ_s are distributed independently, identically, and according to the Lévy law (5). We however need to specify the parameter β in the Lévy formula. We do this by comparing our model with observations. The observed scaling of pulse broadening is close to $\tau_d \propto \lambda^4 d^4$, while our model gives $\tau \propto \lambda^4 d^{(2+\beta)/\beta}$, as is seen from the scaling for sums of Lévy-distributed variables, $\sum^m \sigma_s \sim m^{1/\beta}$, following Eq. (5). Thus, we obtain, $\beta \approx 2/3$.

The standard Gaussian model is not able to reproduce both observational scalings, λ^4 and d^4, simultaneously. Our model reproduces them naturally. Moreover, it predicts that the probability distribution function of electron density gradients in the interstellar turbulence decays as $P(\sigma) \sim |\sigma|^{-1-\beta} \sim |\sigma|^{-5/3}$. Numerical simulations demonstrate that compressible turbulence indeed produces power-law distributions of density gradients, with $\beta < 2$ [16]. So far,

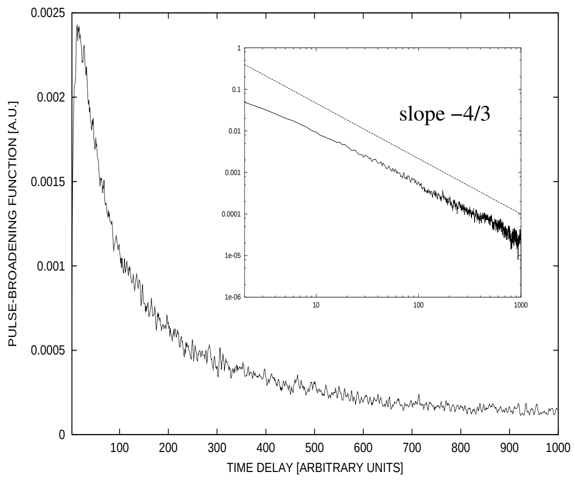

Figure 2: Pulse-broadening functions for the model of linear density fluctuations obeying the Lévy statistics with $\beta = 2/3$. The distribution of the time delay (6) is found numerically using 10^6 rays. The insert shows the large-time asymptotics of the curve in the log-log scale.

the theories of scintillations have exploited only second-order correlators of the density fluctuations, while in our approach these correlators do not exist (or do not matter) and one must work with the whole probability distribution function.

Although our primary goal was to explain the observational scaling relations, our model can predict the shapes of the signals. First, we present a qualitative argument. The delay time is proportional to the square of the typical deflection angle of the ray trajectory, $\tau \propto \theta^2$, where θ has the Lévy distribution $P_\beta(\theta)$. Therefore, the distribution of arrival times is $I(\tau) \propto P_\beta(\tau^{1/2})\tau^{-1/2}$, with the asymptotic form $I(\tau) \propto \tau^{-4/3}$ as $\tau \to \infty$. Fig. (2) shows the distribution of τ from numerical calculation, with a power-law decay at long times, as expected. The power-like decay of these distributions implies that one cannot characterize their widths by their moments. We demonstrated that the observed shape of the scattered pulse is directly related to the *whole* probability distribution of gradients in electron density in the ISM (not to its second moment, as was previously assumed), therefore, observational data offer the new possibility of characterizing interstellar turbulence [9].

The curves in Fig. (2) closely resemble observational signal shapes, however, they are not quite the same. The presented analytical shapes are the result of averaging over an ensemble of density fluctuations, which corresponds to observational average over infinite time. In practice, the averaging time is small, so the long tail of the distribution, which is dominated by rare events, may not be resolved. Also, observational shape of the signal is a convolution of the pulse-broadening function with the intrinsic pulsar signal and with the device resolution curve [9]. We do not consider the last two factors here. Another essential case is a non-analytic density field, which is more natural for turbulent cascades [12, 11, 17]. We will analyze these effects in future work.

Finally, we would like to comment on the original explanation of the anomalous d-scaling by Sutton [5]. Sutton suggested that encounters with much more-strongly-scattering HII regions become more probable on longer lines of sight. This however requires a perhaps surprisingly close coordination of DM (over 1.7 orders of magnitude) with τ_d (over 8 orders of magnitude). Sutton's proposal assumes essentially *non-stationary*, in z direction, statistics of density distribution. Our proposal also invokes rare, large events, but in a statistically *stationary* way.

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