

Homework Two Solutions

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1 Problem One

1.1 Part a

To show that this operator is not Hermitian, we will show that it *fails* to satisfy the equation

$$\langle f|\hat{D}|g\rangle = \langle g|\hat{D}|f\rangle^*, \quad (1)$$

which is one of the ways to state the Hermiticity of an operator D .

Now, in this particular case, we have

$$\langle f|\hat{D}|g\rangle = \int_{-\infty}^{\infty} f^*(x) \left(\frac{dg}{dx} \right) dx, \quad (2)$$

along with,

$$\langle g|\hat{D}|f\rangle = \int_{-\infty}^{\infty} g^*(x) \left(\frac{df}{dx} \right) dx. \quad (3)$$

So we see that we want to try to investigate a possible relation between two integrals, which are almost the same, but differ in terms of which term has a derivative acting on it (along with the complex conjugation on the other term). Because of this, we suspect that we may want to use integration by parts.

Using integration by parts on the first integral, we see that

$$\int_{-\infty}^{\infty} f^*(x) \left(\frac{dg}{dx} \right) dx = f^*(x)g(x)|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} g(x) \left(\frac{df^*}{dx} \right) dx. \quad (4)$$

Because f and g are square-integrable, they must vanish at infinity, and so the boundary term disappears, leaving

$$\int_{-\infty}^{\infty} f^*(x) \left(\frac{dg}{dx} \right) dx = - \int_{-\infty}^{\infty} g(x) \left(\frac{df^*}{dx} \right) dx, \quad (5)$$

and because it does not matter in what order we perform the differentiation and complex-conjugation of f (which you should verify for yourself), we arrive at

$$\int_{-\infty}^{\infty} f^*(x) \left(\frac{dg}{dx} \right) dx = - \int_{-\infty}^{\infty} g(x) \left(\frac{df}{dx} \right)^* dx = - \left(\int_{-\infty}^{\infty} g(x)^* \left(\frac{df}{dx} \right) dx \right)^*. \quad (6)$$

We thus arrive at the result

$$\langle f|\hat{D}|g\rangle = -\langle g|\hat{D}|f\rangle^*, \quad (7)$$

which actually says that D is *anti*-Hermitian, and thus not Hermitian. Notice that anti-Hermitian operators still have some nice properties (they are diagonalizable, for example), however, their eigenvalues are all *pure imaginary*, not real.

1.2 Part b

Notice that,

$$\hat{P} = -i\hbar\hat{D}, \quad (8)$$

and so we have,

$$\langle f|\hat{P}|g\rangle = \langle f|(-i\hbar\hat{D})|g\rangle = -i\hbar\langle f|\hat{D}|g\rangle. \quad (9)$$

Now, if we use the result from part a, we arrive at

$$\langle f|\hat{P}|g\rangle = -i\hbar\langle f|\hat{D}|g\rangle = i\hbar\langle g|\hat{D}|f\rangle^* = (-i\hbar)^*\langle g|\hat{D}|f\rangle^*, \quad (10)$$

and if we bring the factor into the inner product, we get,

$$\langle f|\hat{P}|g\rangle = \langle g|-i\hbar\hat{D}|f\rangle^* = \langle g|\hat{P}|f\rangle^*. \quad (11)$$

Thus, the momentum operator is indeed Hermitian. Notice that this result shows that multiplying an anti-Hermitian operator by a factor of i turns it into a Hermitian operator. The reverse is also true.

1.3 Part c

We have,

$$\langle f|\hat{P}^2|g\rangle = \langle f|\hat{P}\hat{P}|g\rangle = \left(\langle f|\hat{P}\right)\left(\hat{P}|g\rangle\right). \quad (12)$$

Now, recall that from the definition of the adjoint of an operator, we have,

$$|a\rangle = \hat{P}|g\rangle \Rightarrow \langle a| = \langle g|\hat{P}^\dagger. \quad (13)$$

This allows us to write,

$$\left(\langle f|\hat{P}\right)\left(\hat{P}|g\rangle\right) = \left[\left(\langle g|\hat{P}^\dagger\right)\left(\hat{P}^\dagger|f\rangle\right)\right]^* = \langle g|\hat{P}^\dagger\hat{P}^\dagger|f\rangle^*, \quad (14)$$

where the complex conjugation comes from the fact that we are reversing the order of an inner product (and notice that the Hermitian conjugation of the operators is important when we take a ket and send it to its dual bra). However, because the momentum operator is Hermitian, we can finally write this as,

$$\langle f|\hat{P}^2|g\rangle = \langle f|\hat{P}\hat{P}|g\rangle = \langle g|\hat{P}^\dagger\hat{P}^\dagger|f\rangle^* = \langle g|\hat{P}\hat{P}|f\rangle^* = \langle g|\hat{P}^2|f\rangle^*, \quad (15)$$

which shows that the square of the momentum operator is also Hermitian. Now, the kinetic energy of a particle in terms of its momentum is

$$K = \frac{p^2}{2m}, \quad (16)$$

and thus we take,

$$\hat{K} = \frac{\hat{P}^2}{2m}. \quad (17)$$

We see that, using the fact that the square of the momentum operator is Hermitian,

$$\langle f|\frac{\hat{P}^2}{2m}|g\rangle = \frac{1}{2m}\langle f|\hat{P}^2|g\rangle = \frac{1}{2m}\langle g|\hat{P}^2|f\rangle^* = \langle g|\frac{\hat{P}^2}{2m}|f\rangle^*, \quad (18)$$

or,

$$\langle f|\hat{K}|g\rangle = \langle g|\hat{K}|f\rangle^*, \quad (19)$$

which shows that the kinetic energy operator is indeed Hermitian.

More generally, we can show that any real-valued function of a Hermitian operator is again a Hermitian operator. Suppose that $f(x)$ is some real-valued function of x which has a Taylor series,

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n, \quad (20)$$

where α_n is real (do NOT confuse this function with the square-integrable functions which are members of the vector space we were previously talking about). By analogy, for a Hermitian operator (or for that matter any other type of operator), we define,

$$f(\hat{H}) = \sum_{n=0}^{\infty} \alpha_n \hat{H}^n. \quad (21)$$

Now, notice that if the n^{th} power of H is Hermitian, then we have for any kets $|a\rangle$ and $|b\rangle$,

$$\langle a|\hat{H}^{n+1}|b\rangle = \left(\langle a|\hat{H}^n\right) \left(\hat{H}|b\rangle\right) = \left(\langle b|\hat{H}\right) \left(\hat{H}^n|a\rangle\right)^* = \langle b|\hat{H}^{n+1}|a\rangle^*, \quad (22)$$

which shows that the next power of H is also Hermitian (in the second equality I left out the dagger symbols because we know that H and its n^{th} power are Hermitian). Thus, because we know the first power of H is Hermitian, and because a given power of H being Hermitian implies that the next highest power is Hermitian, we see that every integer power of H must be Hermitian.

Using this fact, we see that,

$$\begin{aligned} \langle a|f(\hat{H})|b\rangle &= \langle a|\sum_{n=0}^{\infty} \alpha_n \hat{H}^n|b\rangle = \sum_{n=0}^{\infty} \alpha_n \langle a|\hat{H}^n|b\rangle = \\ &= \sum_{n=0}^{\infty} \alpha_n \langle b|\hat{H}^n|a\rangle^* = \langle b|\sum_{n=0}^{\infty} \alpha_n \hat{H}^n|a\rangle^* = \langle b|f(\hat{H})|a\rangle^* \end{aligned}, \quad (23)$$

showing that indeed, any real-valued function of a Hermitian operator is itself Hermitian.

This result is important for the internal consistency of Quantum Mechanics. In Quantum Mechanics, observables are associated with Hermitian operators. Clearly, if I'm able to measure some quantity, I'm able to measure any function of that quantity (since I can just explicitly compute the value of the function), and so any function of a Hermitian operator must yield another Hermitian operator for this scheme to work.

2 Problem Two

2.1 Part a

Suppose we have an operator H which is real and symmetric. Because it is a real matrix, we have,

$$H_{ij} = H_{ij}^*, \quad (24)$$

and because H is a Hermitian matrix, we also have,

$$H_{ij} = H_{ji}^*. \quad (25)$$

If we combine these results, we see that

$$H_{ij} = H_{ij}^* = H_{ji}, \quad (26)$$

where the equality of the first and last expressions is the definition of a symmetric matrix.

2.2 Part b

If U is our matrix in question, then because it is real-valued, we have,

$$(U^T)_{ij} = U_{ji} = U_{ji}^* = (U^\dagger)_{ij}, \quad (27)$$

where the first equality is the definition of the transpose, the second equality is from the fact that the matrix is real-valued, and the third equality is the definition of the transpose conjugate. So in other words, when U is real-valued, the transpose is the same as the transpose conjugate, or,

$$U^T = U^\dagger. \quad (28)$$

Also, because U is unitary, we have,

$$U^{-1} = U^\dagger. \quad (29)$$

Combining the above results, we have

$$U^T = U^{-1}, \quad (30)$$

which is the desired result.

2.3 Part c

Part c can be derived from the result of part d, so in part d I will show part c

2.4 Part d

Recall that the adjoint of an operator is defined through the dual correspondence

$$\langle a | \hat{U}^\dagger \leftrightarrow \hat{U} | a \rangle, \quad (31)$$

which is to say that if

$$|c\rangle = \hat{U} |a\rangle, \quad (32)$$

then,

$$\langle c | = \langle a | \hat{U}^\dagger. \quad (33)$$

Notice that we often have the notation for the dual correspondence

$$\langle \hat{U} a | \leftrightarrow \hat{U} | a \rangle, \quad (34)$$

which is to say that,

$$\langle \hat{U} a | \equiv \langle a | \hat{U}^\dagger. \quad (35)$$

As a result, we have

$$\langle \hat{U} a | \hat{U} b \rangle = \langle a | \hat{U}^\dagger \hat{U} | b \rangle. \quad (36)$$

However, because U is unitary, we have

$$\langle a|\hat{U}^\dagger\hat{U}|b\rangle = \langle a|\hat{U}^{-1}\hat{U}|b\rangle = \langle a|\hat{I}|b\rangle = \langle a|b\rangle, \quad (37)$$

and so indeed,

$$\langle \hat{U}a|\hat{U}b\rangle = \langle a|b\rangle. \quad (38)$$

Using this result, we can derive the result of part c. Using the definition of the norm in terms of the inner product, we have

$$\|\hat{U}|a\rangle\| = \sqrt{\langle \hat{U}a|\hat{U}a\rangle} = \sqrt{\langle a|a\rangle} = \| |a\rangle \|, \quad (39)$$

which is the desired result for part c.

2.5 Part e

There are two ways to do this problem. The first way is to use the result

$$(AB)^\dagger = B^\dagger A^\dagger, \quad (40)$$

for any two operators A and B . To see this, suppose we have

$$|b\rangle = (AB)|a\rangle, \quad (41)$$

for some kets $|a\rangle$ and $|b\rangle$. If we write,

$$|c\rangle \equiv B|a\rangle, \quad (42)$$

then we have

$$|b\rangle = A|c\rangle, \quad (43)$$

which implies

$$\langle b| = \langle c|A^\dagger. \quad (44)$$

Now, because we also have

$$|c\rangle \equiv B|a\rangle \Rightarrow \langle c| = \langle a|B^\dagger, \quad (45)$$

then we see that

$$\langle b| = \langle c|A^\dagger \Rightarrow \langle b| = (\langle a|B^\dagger)A^\dagger = \langle a|B^\dagger A^\dagger. \quad (46)$$

By comparing the two equations

$$\begin{aligned} |b\rangle &= (AB)|a\rangle \\ \langle b| &= \langle a|B^\dagger A^\dagger \end{aligned} \quad (47)$$

we see that the adjoint of (AB) is indeed given by the previously proposed formula. If we use this, and recall that H is Hermitian while U is Unitary, we see that

$$H'^\dagger = (U^{-1}(HU))^\dagger = (HU)^\dagger (U^{-1})^\dagger = U^\dagger H^\dagger (U^\dagger)^\dagger = U^{-1}HU = H', \quad (48)$$

and thus we see that H' is indeed Hermitian.

The other way we can derive this result is to attempt to verify the equality

$$\langle a|H'|b\rangle = \langle b|H'|a\rangle^* \quad (49)$$

for any kets $|a\rangle$ and $|b\rangle$, by writing H' in terms of U and H and then using properties of these two original operators. I'll leave this method for people to work out on their own (feel free to ask me if you have questions).

2.6 Part f

We want to verify that

$$\hat{U} = e^{i\hat{H}} \quad (50)$$

is a unitary operator, where

$$e^{i\hat{H}} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (i\hat{H})^n. \quad (51)$$

First, recall that if A and B commute with each other, then

$$e^A e^B = e^{A+B}, \quad (52)$$

which can be derived as a result of the BakerCampbellHausdorff formula. Clearly, $i\hat{H}$ commutes with $-i\hat{H}$, and so we have

$$e^{i\hat{H}} e^{-i\hat{H}} = e^{i\hat{H}-i\hat{H}} = e^0 = \hat{I}, \quad (53)$$

from which it follows that

$$(e^{i\hat{H}})^{-1} = e^{-i\hat{H}} \Rightarrow \hat{U}^{-1} = e^{-i\hat{H}}. \quad (54)$$

To find the adjoint of $\hat{U}$, we have

$$(e^{i\hat{H}})^\dagger = \left(\sum_{n=0}^{\infty} \frac{1}{n!} (i\hat{H})^n \right)^\dagger = \sum_{n=0}^{\infty} \left(\frac{1}{n!} (i\hat{H})^n \right)^\dagger, \quad (55)$$

since taking the adjoint is a linear operation. Now, I propose that

$$\left(\frac{1}{n!} (i\hat{H})^n \right)^\dagger = \frac{1}{n!} (-i\hat{H})^n, \quad (56)$$

which we can prove by induction. First, for the case of $n = 1$, notice that

$$(i\hat{H})^\dagger = (i\hat{I}\hat{H})^\dagger = \hat{H}^\dagger (i\hat{I})^\dagger. \quad (57)$$

Now, clearly,

$$(i\hat{I})^\dagger = -i\hat{I}, \quad (58)$$

since if we have a matrix which is just a bunch of diagonal entries which are all i , then the conjugate transpose will do nothing other than multiply all of the diagonal components by -1 (can you show this result for operators in general, without resorting to matrices?). Thus, we find

$$(i\hat{H})^\dagger = \hat{H}^\dagger (i\hat{I})^\dagger = \hat{H} (-i\hat{I}) = -i\hat{H}\hat{I} = -i\hat{H}, \quad (59)$$

which verifies the case $n = 1$. Note that we used the fact that $\hat{H}$ is Hermitian. Now, notice that

$$\left(\frac{1}{(n+1)!} (i\hat{H})^{n+1} \right)^\dagger = \left(\frac{1}{n!} (i\hat{H})^n \frac{1}{(n+1)} (i\hat{H}) \right)^\dagger = \left(\frac{1}{(n+1)!} i\hat{H} \right)^\dagger \left(\frac{1}{n!} (i\hat{H})^n \right)^\dagger, \quad (60)$$

where I broke the term into two pieces, and then used the rule for finding the adjoint of a product. Now, if the formula holds for the n^{th} case, then we can write

$$\left(\frac{1}{(n+1)}i\hat{H}\right)^\dagger \left(\frac{1}{n!}(i\hat{H})^n\right)^\dagger = \left(\frac{1}{(n+1)}(-i\hat{H})\right) \left(\frac{1}{n!}(-i\hat{H})^n\right) = \frac{1}{(n+1)!}(-i\hat{H})^{n+1}, \quad (61)$$

which shows that the formula holds for the next highest case. Thus, by induction, we have proven that

$$\left(\frac{1}{n!}(i\hat{H})^n\right)^\dagger = \frac{1}{n!}(-i\hat{H})^n, \quad (62)$$

With this result, we can write

$$\sum_{n=0}^{\infty} \left(\frac{1}{n!}(i\hat{H})^n\right)^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!}(-i\hat{H})^n \equiv e^{-i\hat{H}}, \quad (63)$$

which means that

$$\hat{U}^\dagger = e^{-i\hat{H}}. \quad (64)$$

Now that we have explicitly found the inverse and adjoint of $\hat{U}$, we see clearly that

$$\hat{U}^\dagger = \hat{U}^{-1}, \quad (65)$$

and so $\hat{U}$ is indeed unitary.

3 Problem Three

3.1 Part a

We have the triple angle formula

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta), \quad (66)$$

which can be found on Wikipedia (or proven for yourself if you are feeling adventurous). Now, if we compute the determinant by expanding along the top row, we have

$$\begin{aligned} \cos(\theta) \cdot \det \begin{bmatrix} 2\cos(\theta) & 1 \\ 1 & 2\cos(\theta) \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 2\cos(\theta) \end{bmatrix} = \\ \cos(\theta)[4\cos^2(\theta) - 1] - [2\cos(\theta)] = 4\cos^3(\theta) - 3\cos(\theta) = \cos(3\theta) \end{aligned} \quad (67)$$

which is the desired result. Don't forget the fact that the signs alternate when we expand along a row or column, with the sign being positive for the upper left entry.

3.2 Part b

To find the transpose, we merely swap rows and columns, which results in

$$A^T = \begin{pmatrix} 0 & -i & 3 \\ 2i & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (68)$$

To find the adjoint, we simply conjugate every element of the transpose, to find

$$A^\dagger = \begin{pmatrix} 0 & i & 3 \\ -2i & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (69)$$

To find the inverse, we can use the technique given in Boas, where we find the cofactors. The formula from Boas is

$$A^{-1} = \frac{1}{\det(A)} C^T ; C_{ij} = \text{cofactor of } A_{ij}. \quad (70)$$

Recall that the cofactor of a matrix element is the value we get when we multiply $(-1)^{i+j}$ by the value of the determinant of the object which is left after we cross out row i and column j . Now, expanding along the last row, the determinant of A is given by

$$\det(A) = 3 \cdot \det \begin{bmatrix} 2i & -1 \\ 2 & 0 \end{bmatrix} = 6. \quad (71)$$

Also, notice that, for example,

$$C_{23} = -\det \begin{bmatrix} 0 & 2i \\ 3 & 0 \end{bmatrix} = 6i \Rightarrow (C^T)_{32} = 6i, \quad (72)$$

so that

$$(A^{-1})_{32} = \frac{1}{\det(A)} (C^T)_{32} = i. \quad (73)$$

Continuing on in this way, we eventually arrive at

$$A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{i}{6} \\ -1 & i & -\frac{1}{3} \end{pmatrix}. \quad (74)$$

Lastly, the complex conjugate is just the matrix where each element is conjugate, which is merely

$$A^* = \begin{pmatrix} 0 & -2i & -1 \\ i & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}. \quad (75)$$

The transpose conjugate is the same as the adjoint (this was a typo in the homework).

I'll leave it to you guys to verify that multiplying A by its inverse gives identity as it should. Feel free to ask questions if you have any confusion over this.

3.3 Part c

An example of a symmetric matrix:

$$\begin{pmatrix} 0 & -2i & 5 \\ -2i & 8 & 6 \\ 5 & 6 & 9 \end{pmatrix} \quad (76)$$

An example of a skew-symmetric matrix:

$$\begin{pmatrix} 0 & 4 & -3 \\ -4 & 0 & 7 \\ 3 & -7 & 0 \end{pmatrix} \quad (77)$$

An example of a real matrix:

$$\begin{pmatrix} 0 & 3 & 8 \\ 1 & -5 & 15 \\ \sqrt{2} & \pi & e \end{pmatrix} \quad (78)$$

An example of a pure imaginary matrix:

$$\begin{pmatrix} i & 0 & -5i \\ 2i & \sqrt{2}i & \pi i \\ 0 & 6i & 0 \end{pmatrix} \quad (79)$$

Please ask me if you have any questions why these matrices are proper examples!

4 Problem Four

4.1 Part a

I'll demonstrate how this works for the first matrix, and then leave the rest up to you. Using the usual formula for matrix multiplication, we have

$$(\Lambda_\phi \cdot \Lambda_\phi^T)_{ij} = \sum_k (\Lambda_\phi)_{ik} (\Lambda_\phi^T)_{kj} = \sum_k (\Lambda_\phi)_{ik} (\Lambda_\phi)_{jk}, \quad (80)$$

which allows us to compute, for example,

$$(\Lambda_\phi \cdot \Lambda_\phi^T)_{12} = \sum_k (\Lambda_\phi)_{1k} (\Lambda_\phi)_{2k} = (\Lambda_\phi)_{11} (\Lambda_\phi)_{21} + (\Lambda_\phi)_{12} (\Lambda_\phi)_{22} + (\Lambda_\phi)_{13} (\Lambda_\phi)_{23}, \quad (81)$$

which is

$$-\cos(\phi)\sin(\phi) + \sin(\phi)\cos(\phi) + 0 \cdot 0 = 0, \quad (82)$$

as it should be. In this way, we can compute each and every component of the product, and verify that we get the identity matrix.

4.2 Part b

Again, we have the matrix multiplication formula

$$(\Lambda_\theta \Lambda_\phi)_{ij} = \sum_k (\Lambda_\theta)_{ik} (\Lambda_\phi)_{kj}. \quad (83)$$

So, for example,

$$(\Lambda_\theta \Lambda_\phi)_{11} = \sum_k (\Lambda_\theta)_{1k} (\Lambda_\phi)_{k1} = (\Lambda_\theta)_{11} (\Lambda_\phi)_{11} + (\Lambda_\theta)_{12} (\Lambda_\phi)_{21} + (\Lambda_\theta)_{13} (\Lambda_\phi)_{31}, \quad (84)$$

which gives

$$(\Lambda_\theta \Lambda_\phi)_{11} = 1 \cdot \cos(\phi) + 0 \cdot (-\sin(\phi)) + 0 \cdot 0 = \cos(\phi). \quad (85)$$

Proceeding in this way, we eventually arrive at

$$\Lambda_\theta \Lambda_\phi = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\cos(\theta) \sin(\phi) & \cos(\phi) \cos(\theta) & \sin(\theta) \\ \sin(\phi) \sin(\theta) & -\cos(\phi) \sin(\theta) & \cos(\theta) \end{pmatrix}. \quad (86)$$

To compute the inverse, we can again use the method in Boas where we find the cofactors (please see problem three for the full expression for the inverse). If we expand along any one of the rows or columns of the product matrix, and use a few simple trig. identities, we quickly come to the conclusion that the determinant of this product matrix is equal to one. So then the inverse will be given by

$$(\Lambda_\theta \Lambda_\phi)^{-1} = C^T ; C_{ij} = \text{cofactor of } (\Lambda_\theta \Lambda_\phi)_{ij}. \quad (87)$$

For example, we have

$$C_{11} = \det \begin{bmatrix} \cos(\phi) \cos(\theta) & \sin(\theta) \\ -\cos(\phi) \sin(\theta) & \cos(\theta) \end{bmatrix} = \cos(\phi) \cdot (\cos^2(\theta) - \sin^2(\theta)) = \cos(\phi), \quad (88)$$

and thus,

$$\left((\Lambda_\theta \Lambda_\phi)^{-1} \right)_{11} = (C^T)_{11} = C_{11} = \cos(\phi). \quad (89)$$

Continuing in this manner, we eventually arrive at the final result

$$(\Lambda_\theta \Lambda_\phi)^{-1} = \begin{pmatrix} \cos(\phi) & -\cos(\theta) \sin(\phi) & \sin(\phi) \sin(\theta) \\ \sin(\phi) & \cos(\phi) \cos(\theta) & -\cos(\phi) \sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}. \quad (90)$$

4.3 Part c

Notice that, if we work out the transpose of the product matrix, we get

$$(\Lambda_\theta \Lambda_\phi)^T = \begin{pmatrix} \cos(\phi) & -\cos(\theta) \sin(\phi) & \sin(\phi) \sin(\theta) \\ \sin(\phi) & \cos(\phi) \cos(\theta) & -\cos(\phi) \sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad (91)$$

which is the same as its inverse. Thus, the product matrix is *also* an orthogonal matrix, and is thus equivalent to a rotation around some other axis.

4.4 Part d

Again, we have the formula

$$(\Lambda_\psi (\Lambda_\theta \Lambda_\phi))_{ij} = \sum_k (\Lambda_\psi)_{ik} (\Lambda_\theta \Lambda_\phi)_{kj}. \quad (92)$$

By now, I assume you guys get the basic idea of how to do this, and so I will quote that the result is

$$\Lambda_\psi (\Lambda_\theta \Lambda_\phi) = \Lambda_\psi \Lambda_\theta \Lambda_\phi =$$

$$\begin{pmatrix} \cos(\phi) \cos(\psi) - \cos(\theta) \sin(\phi) \sin(\psi) & \cos(\psi) \sin(\phi) + \cos(\phi) \cos(\theta) \sin(\psi) & \sin(\psi) \sin(\theta) \\ -\cos(\psi) \cos(\theta) \sin(\phi) - \cos(\phi) \sin(\psi) & \cos(\phi) \cos(\psi) \cos(\theta) - \sin(\phi) \sin(\psi) & \cos(\psi) \sin(\theta) \\ \sin(\phi) \sin(\theta) & -\cos(\phi) \sin(\theta) & \cos(\theta) \end{pmatrix}. \quad (93)$$

Please feel free to ask me questions if you have any trouble reproducing this result.

5 Problem Five

5.1 Part a

To find the eigenvalues, let's start with the characteristic equation,

$$\det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & c \\ c & b - \lambda \end{pmatrix} = 0 \quad (94)$$

which leads to,

$$(a - \lambda)(b - \lambda) - c^2 = \lambda^2 - (a + b)\lambda + (ab - c^2) = 0. \quad (95)$$

Solving this equation for λ , we get

$$\lambda_{\pm} = \frac{1}{2} \left(a + b \pm \sqrt{(a + b)^2 - 4(ab - c^2)} \right), \quad (96)$$

or,

$$\lambda_{\pm} = \frac{1}{2} \left(a + b \pm \sqrt{(a - b)^2 + 4c^2} \right), \quad (97)$$

after rearranging. Notice that because the term under the radical can never be negative (since a, b, and c are real), these eigenvalues will always be real. This is to be expected, since a real, symmetric matrix is Hermitian. If a = b and c = 0, then the eigenvalues will be degenerate. In this case, the matrix will be a multiple of the identity operator.

To find the eigenvectors, we write,

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \lambda_{\pm} \begin{pmatrix} f \\ g \end{pmatrix} \quad (98)$$

which leads to the set of equations

$$\begin{pmatrix} fa + gc \\ fc + gb \end{pmatrix} = \begin{pmatrix} \lambda_{\pm} f \\ \lambda_{\pm} g \end{pmatrix}. \quad (99)$$

Now, this set of equations is redundant, and we need only work with one of them. To see why, notice that if λ is an eigenvalue of some operator A, with $|\lambda\rangle$ being the associated eigenvector, then we have,

$$\hat{A}|\lambda\rangle = \lambda|\lambda\rangle, \quad (100)$$

and if we multiply both sides by some number β , we get, after pulling through the factor,

$$\hat{A}(\beta|\lambda\rangle) = \lambda(\beta|\lambda\rangle), \quad (101)$$

which shows that $\beta|\lambda\rangle$ is also an eigenvector, with the same eigenvalue. So the eigenvalue equation doesn't determine the norm of the eigenvector, and so we should expect that the set of equations is under-determined. If we take the first equation, what we find is

$$f = -\frac{gc}{a - \lambda_{\pm}}. \quad (102)$$

However, we should be careful to verify that $\lambda_{\pm} \neq a$, or else we would be dividing by zero in this expression. For this to be true, we would need to have,

$$a = \frac{1}{2} \left(a + b \pm \sqrt{(a-b)^2 + 4c^2} \right), \quad (103)$$

or,

$$\pm \sqrt{(a-b)^2 + 4c^2} = \frac{1}{2}(a-b). \quad (104)$$

If we square both sides of this expression, we get, after rearranging,

$$\frac{3}{4}(a-b)^2 + 4c^2 = 0. \quad (105)$$

The only way this can be true is if $a = b$ and $c = 0$, in which case we have a multiple of the identity matrix. I will return to this special case later, and for now assume that this is not true. Notice that this equation is not so useful if $c = 0$, regardless of the value of a or b . I'll also come back to this case in a moment.

For now, if we take $g = 1$, the eigenvectors are, up to normalization,

$$|\lambda_{\pm}\rangle \mapsto \begin{pmatrix} 1 \\ \frac{c}{\lambda_{\pm} - a} \end{pmatrix}. \quad (106)$$

The normalization is of course up to us to choose. Assuming that the matrix is the representation of an operator in an orthonormal basis, and there is an inner product defined, we can compute the inner product as

$$\langle \lambda_+ | \lambda_- \rangle = \left(1, \frac{c}{\lambda_+ - a} \right) \begin{pmatrix} 1 \\ \frac{c}{\lambda_- - a} \end{pmatrix} = 1 + \frac{c^2}{(\lambda_+ - a)(\lambda_- - a)}. \quad (107)$$

Now,

$$\begin{aligned} & (\lambda_+ - a)(\lambda_- - a) = \\ & \left(\frac{1}{2}(a+b + \sqrt{(a-b)^2 + 4c^2}) - a \right) \left(\frac{1}{2}(a+b - \sqrt{(a-b)^2 + 4c^2}) - a \right) \end{aligned} \quad (108)$$

Upon simplification of this result, we find that

$$(\lambda_+ - a)(\lambda_- - a) = -c^2, \quad (109)$$

so that,

$$\langle \lambda_+ | \lambda_- \rangle = 1 + \frac{c^2}{(\lambda_+ - a)(\lambda_- - a)} = 1 - \frac{c^2}{c^2} = 0 \quad (110)$$

as expected. Notice that even if we had not defined any inner product on our space, we could still just take the above result to be the definition of what we mean by perpendicular.

Now, what if $c = 0$, without $a = b$? Well, in that case, we have,

$$\begin{aligned}\lambda_+ &= \frac{1}{2}(a + b + \sqrt{(a - b)^2}) = a, \\ \lambda_- &= \frac{1}{2}(a + b - \sqrt{(a - b)^2}) = b.\end{aligned}\tag{111}$$

Thus, we have, for one of the eigenvectors,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = a \begin{pmatrix} f \\ g \end{pmatrix},\tag{112}$$

which leads to,

$$\begin{pmatrix} fa \\ gb \end{pmatrix} = \begin{pmatrix} fa \\ ga \end{pmatrix}.\tag{113}$$

This implies then that g must be zero (since a and b are not equal), and thus, up to normalization,

$$|\lambda_+\rangle \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}.\tag{114}$$

With a similar approach, we also arrive at

$$|\lambda_-\rangle \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\tag{115}$$

These two vectors are quite clearly perpendicular.

Now, lastly, what if $c = 0$ and $a = b$? Well, in this case, A is a multiple of the identity operator,

$$\hat{A} = \lambda \hat{I},\tag{116}$$

and so it follows that, for *any* vector $|x\rangle$,

$$\hat{A}|x\rangle = \lambda \hat{I}|x\rangle = \lambda|x\rangle.\tag{117}$$

Thus, in this very special case, *every* vector is an eigenvector of A . Clearly in this case, it is no longer true that the eigenvectors are perpendicular. However, as a practical matter, this case doesn't come up very often.

5.2 Part b

If we denote our matrix by A , then the eigenvalue equation reads,

$$\hat{A}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle,\tag{118}$$

where i is an index that labels which eigenvector we have. If the dimensionality of A is n , then we know that because A is Hermitian, it must have n eigenvectors, whose vectors constitute a complete basis. Now, because A is Hermitian,

$$\hat{A} = \hat{A}^\dagger,\tag{119}$$

and because A is Unitary,

$$\hat{A}^{-1} = \hat{A}^\dagger, \quad (120)$$

so therefore,

$$\hat{A} = \hat{A}^{-1}, \quad (121)$$

or,

$$\hat{A}^2 = \hat{I}. \quad (122)$$

If we apply A to both sides of the eigenvalue equation, we have,

$$\hat{A} \left(\hat{A} |\lambda_i\rangle \right) = \hat{A} (\lambda_i |\lambda_i\rangle), \quad (123)$$

and using the fact that A is equal to its own inverse, along with the fact that $|\lambda_i\rangle$ is an eigenvector of A, this becomes,

$$|\lambda_i\rangle = \lambda_i \left(\hat{A} |\lambda_i\rangle \right) = \lambda_i^2 |\lambda_i\rangle, \quad (124)$$

which thus leads to

$$\lambda_i^2 = 1 \Rightarrow \lambda_i = \pm 1. \quad (125)$$

5.3 Part c

The eigenvalue equation reads

$$\hat{C} |x_i\rangle = \lambda_i |x_i\rangle. \quad (126)$$

Aside from knowing that C has an inverse and at least some eigenvectors, we do not know whether it is diagonalizable, or whether it forms a complete basis. So we should not assume any of these things.

If we multiply both sides of the eigenvalue equation by the inverse of C, we arrive at,

$$|x_i\rangle = \lambda_i \left(\hat{C}^{-1} |x_i\rangle \right) \Rightarrow \hat{C}^{-1} |x_i\rangle = \frac{1}{\lambda_i} |x_i\rangle, \quad (127)$$

assuming that $\lambda_i \neq 0$. Could λ_i be equal to zero? Well, if it were, we would have

$$\hat{C} |x_i\rangle = 0, \quad (128)$$

and if we apply the inverse of C to both sides, we get,

$$|x_i\rangle = \hat{C}^{-1}(0). \quad (129)$$

Now, because the inverse of C is also a matrix, and any matrix operator is a linear operator, the inverse of C must be a linear operator. Notice that for *any* vector $|a\rangle$, and *any* linear operator T, we must have

$$\hat{T}|a\rangle = \hat{T}(|a\rangle + 0) = \hat{T}|a\rangle + \hat{T}(0) \Rightarrow \hat{T}(0) = 0, \quad (130)$$

or that the action of any linear operator on the zero vector must return back the zero vector. This would then imply that for an eigenvalue of zero,

$$|x_i\rangle = \hat{C}^{-1}(0) = 0. \quad (131)$$

However, the zero vector cannot be an eigenvector (because we exclude it from consideration), and so the above statement is impossible. Thus, we conclude that an invertible matrix cannot have a zero eigenvalue, and thus we need not worry about the possibility of λ_i being equal to zero, and we have in all cases,

$$\hat{C}^{-1}|x_i\rangle = \frac{1}{\lambda_i}|x_i\rangle, \quad (132)$$

which shows that $|x_i\rangle$ is also an eigenvector of $\hat{C}^{-1}$, with the reciprocal eigenvalue.

But have we found *all* of the eigenvectors for $\hat{C}^{-1}$? To see that we have, suppose that $\hat{C}^{-1}$ had at least one more eigenvector, labeled as $|q_i\rangle$. Because we have

$$\left(\hat{C}^{-1}\right)^{-1} = \hat{C}, \quad (133)$$

then we could go through the above argument again, this time applied to $\hat{C}^{-1}$, to show that $|q_i\rangle$ must also be an eigenvector of $\hat{C}$. Thus, the sets of eigenvectors for $\hat{C}$ and its inverse *must* coincide, and so we have indeed found all of them.

Thus, in summary, the eigenvectors of $\hat{C}^{-1}$ are the same as those for $\hat{C}$, and the eigenvalues are the reciprocals of those for $\hat{C}$.