

Homework 3 - Solutions

Problem 1 - Integer Quantum Hall Effect

The wavefunction of the “free” 2d electrons in a magnetic field (choosing the gauge $\vec{A} = By\hat{x}$) are given by:

$$\psi(x, y) = e^{ik_x x} Y(y) \quad (1)$$

$$\Rightarrow \frac{1}{2m} \left[-\hbar^2 \frac{d^2}{dy^2} + (eB)^2 \left(y - \frac{\hbar k_x}{eB} \right)^2 \right] Y = \varepsilon Y \quad (2)$$

$$\Rightarrow \text{a 1d harmonic oscillator with frequency } \omega_c \text{ and center } y_0 \text{ given by} \quad (3)$$

$$\omega_c = \frac{eB}{c} \quad \text{and center } y_0 = \frac{\hbar k_x}{eB} = k_x \ell^2, \quad \ell = \sqrt{\frac{\hbar}{eB}}. \quad (4)$$

Then, the energy levels are given by $\varepsilon_n = \hbar\omega_c(n + \frac{1}{2})$ and are independant of k_x . Thus, the degeneracy of a Landau level is given by the number of possible k_x states. For a sample of length L_x we have

$$k_x = \frac{2\pi}{L_x} i, \quad i = 0, 1, 2, \dots \quad (5)$$

But, now there is a restriction on the value of y_0 , since the system has a finite size in the y direction then $0 < y_0 < L_y$. So we have

$$0 < y_0 = k_x \ell^2 < L_y \quad (6)$$

Then, since $k_x = 2\pi i/L_x$ we can find that

$$k_x \ell^2 < L_y \Rightarrow \frac{2\pi \ell^2}{L_x} i < L_y \quad (7)$$

$$\Rightarrow 0 < i < \frac{L_x L_y}{2\pi \ell^2} \quad (8)$$

Thus, since the system is spin polarized, the number of states in one degenerate Landau level is the number of k_x states, which is equal to

$$N_\phi = \frac{L_x L_y}{2\pi \ell^2}. \quad (9)$$

Then, a system with electron density $n = N/V$ contains exactly $N = nL_x L_y$ states. So, if we want to fill exactly k Landau levels we need

$$k = \frac{N}{N_\phi} = nL_x L_y \frac{2\pi \ell^2}{L_x L_y} = 2\pi \ell^2 n = 2\pi n \frac{\hbar}{eB} \quad (10)$$

Therefore, to fill exactly k Landau levels we need

$$B = \frac{2\pi \hbar n}{ek} = \frac{\hbar n}{ek} \quad (11)$$

Problem 2 - Topological Insulators

(a)

Let's start with the Hamiltonian for graphene

$$\begin{pmatrix} \varepsilon_0 & f(k) \\ f^*(k) & \varepsilon_0 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad (12)$$

$$\text{where } f(k) = -\gamma \left[e^{-ik_y a} + 2 \cos\left(\frac{\sqrt{3}k_x a}{2}\right) e^{ik_y a/2} \right] \quad (13)$$

$$\text{so } f(K+q) \approx \frac{3}{2}\gamma a(q_x + iq_y) \quad \text{and} \quad f(K'+q) \approx \frac{-3}{2}\gamma a(q_x - iq_y) \quad (14)$$

Now, we want to include the effects of spin-orbit interactions by adding a term to the Hamiltonian

$$H_{\text{SO}} \psi_{R\varepsilon A}^\sigma = i\sigma\lambda \sum_{i=1}^3 (\psi_{R+e_i-e_{i+1}}^\sigma - \psi_{R+e_i-e_{i-1}}^\sigma) \quad (15)$$

$$H_{\text{SO}} \psi_{R\varepsilon B}^\sigma = i\sigma\lambda \sum_{i=1}^3 (\psi_{R-e_i+e_{i+1}}^\sigma - \psi_{R-e_i+e_{i-1}}^\sigma) \quad (16)$$

Assume a solution of the form $\psi = \psi_{A,B} e^{i\vec{k}\cdot\vec{R}}$ and sub into H_{SO} above to get

$$H_{\text{SO}} \psi_{R\varepsilon A}^\sigma = \varepsilon \psi_A^\sigma e^{i\vec{k}\cdot\vec{R}} = i\sigma\lambda \sum_{i=1}^3 \left(\psi_A e^{i\vec{k}\cdot R+e_i-e_{i+1}} - \psi_A e^{i\vec{k}\cdot R+e_i-e_{i-1}} \right) \quad (17)$$

This then leads to

$$\varepsilon_{\text{SO}} = \Delta^\sigma(k) \quad (18)$$

$$\Delta^\uparrow(k) = -\Delta^\downarrow(k) = \Delta(k) = -2\lambda \sum_{i=1}^3 \sin(\vec{k}\cdot(e_i - e_{i+1})) \quad (19)$$

Adding this to the usual Hamiltonian for graphene leads to the energy eigenvalues

$$\varepsilon_\pm^\sigma = \varepsilon_0 \pm \sqrt{|f(k)|^2 + \Delta^2(k)} \quad (20)$$

Now, we want to find the edge modes on the right side of the graphene sample around the K' point in the Brillouin zone. To do this, let's assume that at the edge of the sample there exists a usual band insulator with a staggered potential V . So then, this adds a constant energy contribution $+V$ to the A sublattice sites and $-V$ to the B sublattice sites. We also want to include the contribution of the spin-orbit coupling term to the Hamiltonian. We can then solve the Hamiltonian for each spin direction separately (since there are no terms in the Hamiltonian which couple spin up and spin down terms). So then, for the spin down electrons the contribution to the graphene Hamiltonian is a term $-\Delta(k)$. So then the total Hamiltonian looks like:

$$\begin{pmatrix} \varepsilon_0 + V - \Delta(k) & f(k) \\ f^*(k) & \varepsilon_0 - V + \Delta(k) \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad (21)$$

$$(22)$$

Expanding $f(k)$ around the K' point then gives

$$\begin{pmatrix} \varepsilon_0 + V - \Delta(k) & -\nu(q_x - iq_y) \\ -\nu(q_x + iq_y) & \varepsilon_0 - V + \Delta(k) \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad (23)$$

Note that for the eigenvalues of this Hamiltonian are found by solving $\det[H - \varepsilon] = 0$. this gives $\varepsilon = \varepsilon_0 \pm \sqrt{|f(k)|^2 + (V - \Delta(k))^2}$, so that $\varepsilon(k)$ can only be gapless around the K and K' points when $V - \Delta = 0$.

Now, for a spin down particle around the K' point, $K' = (-\frac{4\pi}{3\sqrt{3}}, 0)$ and $\sin(\vec{e}_i - \vec{e}_{i+1}) = \sin(2\pi/3) \Rightarrow \Delta(K') = 3\sqrt{3}\lambda = +|\Delta|$. Therefore, the value of the gap induced by the combination of spin-orbit coupling and the staggered potential, around the K' point is (recall $\Delta^\downarrow = -\Delta^\uparrow$):

$$\text{gap (spin up)} = V + \Delta^\uparrow(K') = V + |\Delta| \quad (24)$$

$$\text{and gap (spin down)} = V - \Delta^\downarrow(K') = V + |\Delta| \quad (25)$$

$$(26)$$

Therefore, near the K' point, we can only have a zero gap for spin-down electrons.

Near the K' point, we saw in class that we can write the function $|f(k)| = -q_x + iq_y$. We want to write our eigenstates as a function of real space position x and wavevector q_y , so let $q_x \rightarrow -i\partial_x$. Then,

$$\begin{pmatrix} \varepsilon_0 + V - |\Delta(k)| & i\nu(\partial_x + q_y) \\ i\nu(\partial_x - q_y) & \varepsilon_0 - V + |\Delta(k)| \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad (27)$$

As in class, assume a solution of the form

$$\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} g(x) \quad (28)$$

$$\text{with } g(x) = g_0 \exp \left[-\frac{1}{v} \int_0^x V_{\text{eff}}(x') dx' \right] \quad (29)$$

where $V_{\text{eff}}(x) = V - |\Delta(x)|$ is negative for $x < 0$, positive for $x > 0$ and equal to zero at $x = 0$. Therefore, the value of the integral is always positive for any value of x (since the direction of integration changes when $V_{\text{eff}}(x)$ changes signs), so that $g(x)$ is an exponentially decaying function centered around $x = 0$.

Note that

$$\partial_x g(x) = g(x) \partial_x \left[-\frac{1}{v} \int_0^x V_{\text{eff}}(x') dx' \right] = g(x) \left(\frac{-1}{v} V_{\text{eff}}(x) \right) \quad (30)$$

using the chain rule on the exponential function and then using the fundamental theorem of calculus.

Sub our ansatz solution into Eq. (29) to get

$$\begin{aligned} \begin{pmatrix} \varepsilon_0 + V_{\text{eff}} & i\nu(\partial_x + q_y) \\ i\nu(\partial_x - q_y) & \varepsilon_0 - V_{\text{eff}} \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} g(x) &= \begin{pmatrix} \varepsilon_0 + V_{\text{eff}} + \nu \left(\frac{-1}{v} V_{\text{eff}} + q_y \right) \\ i\nu \left(\frac{-1}{v} V_{\text{eff}} + q_y \right) - i(\varepsilon_0 - V_{\text{eff}}) \end{pmatrix} g(x) \\ &= \begin{pmatrix} \varepsilon_0 + \nu q_y \\ -i(\nu q_y) - i(\varepsilon_0) \end{pmatrix} g(x) \\ &= (\varepsilon_0 + \nu q_y) \begin{pmatrix} 1 \\ -i \end{pmatrix} g(x) \end{aligned} \quad (31)$$

Therefore, we have just shown that our ansatz is indeed an eigenstate of the Dirac equation describing our spin-orbit coupled graphene system near the K' point for a spin down electron. Furthermore, we also showed that the eigenvalue of this state is given by

$$\varepsilon = \varepsilon_0 + \nu q_y \quad (32)$$

This is the same energy we would find if our system contained a right moving dirac fermion (since the energy of such a particle is $v|p|$ for some velocity v and momentum p).

If we look at the form of our ansatz wavefunction, we see that the function $g(x)$ is exponentially localized around the $x = 0$ point which we defined to be on the right edge of our system (at the interface between the spin-orbit coupled graphene and the band insulator).

Therefore, we can say that at the right edge of the system around the K' point there is a spin-down edge state which is right moving.

(b)

We now want to repeat the above calculation, looking at the left edge of the system. In this case, our function $V_{\text{eff}}(x)$ will be changed, since now the band insulator with nonzero parameter V is to the left of our spin-orbit coupled system with nonzero parameter Δ .

We again define $V_{\text{eff}}(x) = V(x) - |\Delta(x)|$, but now we define

$$g(x) = g_0 \exp \left[-\frac{1}{v} \int_x^0 V_{\text{eff}}(x') dx' \right] \quad (33)$$

note the switching of the bounds of integration. Again, now the argument of the exponential is negative for all values of x , so that we have an exponentially decreasing function centered around the left edge of the system. So, when we describe our eigenstate by with function the wavefunction will have a non-negligible probability density only near the left edge of the system.

Now, since we are at the left edge of the system $V_{\text{eff}}(x)$ is greater than zero for $x < 0$ and less than zero for $x > 0$. Therefore

$$\partial_x g(x) = g(x) \partial_x \left[-\frac{1}{v} \int_x^0 V_{\text{eff}}(x') dx' \right] = g(x) \partial_x \left[+\frac{1}{v} \int_0^x V_{\text{eff}}(x') dx' \right] = g(x) \left(\frac{+1}{v} V_{\text{eff}}(x) \right) \quad (34)$$

Now, solve the Dirac equation exactly as in the previous section, only this time, look at spin-up particles near the K -point so that $|f(k)| = +q_x + iq_y$. Then, the dirac equation takes the form

$$\begin{pmatrix} \varepsilon_0 + V - |\Delta(k)| & i\nu(-\partial_x + q_y) \\ i\nu(-\partial_x - q_y) & \varepsilon_0 - V + |\Delta(k)| \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad (35)$$

assume a solution of the form

$$\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} g(x) \quad (36)$$

with $g(x)$ defined above.

Subbing in our ansatz gives

$$\begin{aligned} \begin{pmatrix} \varepsilon_0 + V_{\text{eff}} & i\nu(-\partial_x + q_y) \\ i\nu(-\partial_x - q_y) & \varepsilon_0 - V_{\text{eff}} \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} g(x) &= \begin{pmatrix} \varepsilon_0 + V_{\text{eff}} + \nu(-(\frac{\pm 1}{v})V_{\text{eff}} + q_y) \\ i\nu(-(\frac{\pm 1}{v})V_{\text{eff}} + q_y) - i(\varepsilon_0 - V_{\text{eff}}) \end{pmatrix} g(x) \\ &= \begin{pmatrix} \varepsilon_0 + \nu q_y \\ -i(\nu q_y) - i(\varepsilon_0) \end{pmatrix} g(x) \\ &= (\varepsilon_0 + \nu q_y) \begin{pmatrix} 1 \\ -i \end{pmatrix} g(x) \end{aligned} \quad (37)$$

Therefore, for the same reason as above, this is a spin-up state localized at the left edge of our system with energy given by the eigenvalue $\varepsilon = \varepsilon_0 + \nu q_y$. Therefore, it describes a right-moving particle, which is the opposite of what we found in class for the right edge of the system.