Homework 4 - Solutions

Problem 1 -Vortices in Rotating Superfluids

To start, first notice that the free energy of a system can be written in terms of it's superfluid density n_s and the wavefunction phase θ as

$$F = \int d^3r \left[\frac{c}{2}(n_s - \bar{n}_s)^2 + \frac{\hbar^2 n_s}{2m}(\nabla\theta)^2 \right] \tag{1}$$

and the superfluid velocity is given by

$$\vec{v}_s = \frac{\hbar}{m}\nabla\theta \tag{2}$$

Then, the total velocity of the superfluid is given by the expression

$$\oint \nabla\theta \cdot ds = 2\pi n \qquad n \in \mathbb{Z} \tag{3}$$

$$\Rightarrow \quad \frac{mv_s}{\hbar}(2\pi R) = 2\pi n \tag{4}$$

For a single vortex, this circulation has 'charge' = 1 so that around any single vortex we have

$$\oint_{\text{vortex}} \nabla\theta \cdot ds = 2\pi \tag{5}$$

Then, if we take a path around the circumference of the sample, so that we enclose all n_v vortices which are in the sample we find

$$\oint_{\text{sample}} \nabla\theta \cdot ds = 2\pi n_v \tag{6}$$

where n_v is the total number of vortices in the sample

Now, we can find an expression for the number density of vortices which equals the total number of vortices divided by the cross-sectional area of the system (with $A = \pi R^2$):

$$\frac{mv_sR}{\hbar} = n_v \tag{7}$$

$$v_s = \frac{2\pi R}{\Delta t} = 2\pi R f \tag{8}$$

$$\Rightarrow \frac{2\pi m R^2 f}{\hbar} = n_v \tag{9}$$

$$\Rightarrow \frac{2mAf}{\hbar} = n_v \tag{10}$$

$$\frac{n_V}{A} = \frac{2mf}{\hbar} \tag{11}$$

Therefore

$$\frac{n_v}{A} = \frac{2mf}{\hbar} = \frac{(2)(4*1.66*10^{-27})(1)}{(1.055*10^{-34})} = 1.2509*10^8\text{m}^{-2} = 12588\text{cm}^{-2} \tag{12}$$

The density of liquid helium is $141.2kg/m^3$. Then, for Helium, $n_s = \rho/m = 2.12*10^{22}$ atoms/cm^2

Part (b)

Now for Na atoms rotating at $60Hz$ we have

$$\frac{n_v}{A} = \frac{2mf}{\hbar} = \frac{(2)(23 * 1.66 * 10^{-27})(60)}{(1.055 * 10^{-34})} = 4.34 * 10^{10}\text{m}^{-2} = 4.34 * 10^{6}\text{cm}^{-2} \tag{13}$$

Note that above I used that the atomic mass of sodium is $\sim 23m_p$. Compare this to the number density of atoms $n_s = 4.3 * 10^{14}\text{cm}^{-2}$

Problem 2 - The Kosterlitz-Thouless Transition

From the course notes, it was seen that the energy cost of a vortex is given by

$$E_{\text{vortex}} = \frac{\pi\hbar^2 n_s d}{m} \ln\left(\frac{R}{a}\right) \tag{14}$$

We need to add to the free energy the contribution due to the entropy due to the different possible positions of the vortex. If we assume that the vortex core has radius a, then the number of possible places to place this core in a circular cross-seciontional area $A = \pi R^2$ is $\pi R^2/(\pi a^2)$

$$S = k\ln(\Omega) = k_B \ln(\frac{\pi R^2}{\pi a^2}) = 2k_B \ln\left(\frac{R}{a}\right) \tag{15}$$

One might worry about including some close packing factor for 2d circles, but this will just give an additive constant to the logarithm term which will be negligible as $R \to \infty$.

Therefore

$$F = E_{\text{vortex}} = TS \quad = \quad \frac{\pi\hbar^2 n_s d}{m} \ln\left(\frac{R}{a}\right) - 2kT\ln\left(\frac{R}{a}\right) \tag{16}$$

$$= \left[\frac{\pi\hbar^2 n_s d}{m} - 2k_B T\right] \ln\left(\frac{R}{a}\right) \tag{17}$$

Now, just note that $\rho \approx m \times n_s \quad \Rightarrow \quad n_s = \rho/m$. Then

$$F = \left[\frac{\pi\hbar^2 \rho d}{m^2} - 2k_B T\right] \ln\left(\frac{R}{a}\right) \tag{18}$$

Then, this free energy changes signs when

$$T = \frac{\pi\hbar^2 \rho d}{2m^2 k_B} \tag{19}$$

Subbing in the given values $\rho = 0.14g/cm^3 = 140\text{kg/m}^3$, $m = 4 * 1.66 * 10^{-27}$ and $d = 3\mathring{A} = 3 * 10^{-10}\text{m}$ Then

$$T \approx \frac{\pi(1.055 * 10^{-34})^2(140)(3 * 10^{-10})}{(2)(4 * 1.66 * 10^{-27})^2(1.38 * 10^{-23})} \approx 1.21K \tag{20}$$

Therefore, we can approximate the Kosterlitz-Thouless transition temperature for superfluid He-4 films with thickness of $3\mathring{A}$ to occur at $\sim T_C = 1.21K$

Problem 3 - Flux Quantum

First, notice that the free energy of a superconducting state with vortices is given by the expression (from the course notes):

$$F = \int d^3r \left[n_s^* \frac{p^2}{2m^*} + a(n_s - n_s^{\text{eq}})^2 + \frac{B^2}{2\mu_0} \right] \tag{21}$$

$$= \int d^3r \left[\frac{n_s}{8m} |\hbar\nabla\theta + 2e\vec{A}|^2 + a(n_s - n_s^{\text{eq}})^2 + \frac{B^2}{2\mu_0} \right] \tag{22}$$

Where to go from the first to the second line we wrote $\psi = \sqrt{n_s^*(r)} e^{i\theta(r)}$ and $\vec{p} = \hbar\nabla\theta - q\vec{A}$. Since superconductivity is caused by the formation of cooper pairs, this gives

$$n_s^* = n_s/2 \ \ , \ \ q = -2e \ \ , \ \ m^* = 2m \tag{23}$$

The free energy in Eq. (26) is minimized for

$$\vec{A} = \frac{\hbar}{2e}\nabla\theta \ \ , \ \ B = 0 \ \ , \ \ n_s = n_s^{\text{eq}} \tag{24}$$

Then, the magnetic flux through a vortex is given by the expression

$$\oint A \cdot d\ell = \frac{\hbar}{2e} 2\pi = \frac{h}{2e} = \varphi_0 \tag{25}$$

So, we see that the quantization of the magnetic flux through a vortex is derived from the magnetic potential which minimized the free energy.

But, now if we go back to Eq. (21), and assume that superconductivity is caused by condensation of 4 electrons. Then $n_s^* = n_s/4$, $q = -4e$, $m* = 4m$. Then, in this case

$$F = \int d^3r \left[\frac{n_s}{16m} |\hbar\nabla\theta + 4e\vec{A}|^2 + a(n_s - n_s^{\text{eq}})^2 + \frac{B^2}{2\mu_0} \right] \tag{26}$$

This free energy is then minimized for

$$\vec{A} = \frac{\hbar}{4e}\nabla\theta \ \ , \ \ B = 0 \ \ , \ \ , n_s = n_s^{\text{eq}} \tag{27}$$

Then, the magnetic flux through a vortex is

$$\oint A \cdot d\ell = \frac{\hbar}{4e} 2\pi = \frac{h}{4e} = \varphi \tag{28}$$

Therefore, if in an experiement, the magnetic flux of vortices in a new superconducting material is found to equal $\pm h/4e$, then this would imply that superconductivity in this new material is caused by Bose-Einstein condensation of groups of 4 electrons.