

Homework 6 - Solutions

Problem 1 - Non-Linear M(H)

The Hamiltonian of a Heisenberg ferromagnet in a magnetic field H is given by

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - g\mu_B \sum_i \mathbf{H} \cdot \mathbf{S}_i, \quad J > 0 \quad (1)$$

It was seen in class that in mean field theory we can study the system

$$H = -J \sum_{\langle ij \rangle} \left[\langle \mathbf{S}_i \rangle \cdot \mathbf{S}_j + \mathbf{S}_i \cdot \langle \mathbf{S}_j \rangle - \langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_j \rangle \right] - g\mu_B \sum_i \mathbf{H} \cdot \mathbf{S}_i \quad (2)$$

$$= -zJ \sum_i \left(\mathbf{m} \cdot \mathbf{S}_i + \text{const.} \right) - g\mu_B \sum_i \mathbf{H} \cdot \mathbf{S}_i \quad (3)$$

$$= - \sum_i (zJ\mathbf{m} + g\mu_B\mathbf{H}) \cdot \mathbf{S}_i \quad (4)$$

where $\mathbf{m} = \langle \mathbf{S}_i \rangle$ and z equals the number of nearest neighbour lattice sites.

We saw in class that the magnetization for a spin-1/2 particle interacting with an effective field h is given by the solution to the equation

$$m = \frac{1}{2} \tanh \left[\frac{h}{2kT} \right] \quad (5)$$

We know that the spins will align in the same direction as the magnetic field so that \mathbf{m} and \mathbf{H} point in the same direction.

Then, in our system, the effective field is given by $h = zJm + \zeta g\mu_B |H|$, where $\zeta = \text{sign}(H)$. Therefore, the magnetization is given by the solution to the equation

$$m = \frac{1}{2} \tanh \left[\frac{zJm + \zeta g\mu_B |H|}{2kT} \right] \quad (6)$$

We also saw in class that the critical point for the Heisenberg ferromagnet in mean field theory is at $kT_c = zJ/4$. Subbing this in for our temperature gives

$$m = \frac{1}{2} \tanh \left[\frac{2(zJm + \zeta g\mu_B |H|)}{zJ} \right] = \frac{1}{2} \tanh \left[2m + \frac{2\zeta g\mu_B |H|}{zJ} \right] \quad (7)$$

When H is small, we know that the magnetization m is also small near the critical point T_c , since the magnetization is zero above the critical temperature and the magnetization is a continuous function of temperature for this model. Therefore we can Taylor expand our $\tanh(x) \approx x - x^3/3$ function

$$m = \frac{1}{2} \left[2m + \frac{2\zeta g \mu_B |H|}{zJ} + \frac{1}{3} \left(2m + \frac{2\zeta g \mu_B |H|}{zJ} \right)^3 \right] \quad (8)$$

$$0 = \frac{\zeta g \mu_B |H|}{zJ} + \frac{1}{6} \left(2m + \frac{2\zeta g \mu_B |H|}{zJ} \right)^3 \quad (9)$$

$$\Rightarrow 2m \approx \left(\frac{6\zeta g \mu_B |H|}{zJ} \right)^{1/3} + \mathcal{O}(H) \quad (10)$$

$$m = \text{sign}(H) \frac{1}{2} \left(\frac{6g\mu_B}{zJ} \right)^{1/3} |H|^{1/3} \quad (11)$$

Where above I expanded the equation above to lowest nonvanishing order in both m and H .

Therefore

$$M = g\mu_B m = \text{sign}(H) \frac{g\mu_B}{2} \left(\frac{6g\mu_B}{zJ} \right)^{1/3} |H|^{1/3} \quad (12)$$

Problem 2 - Quantum transverse-field Ising model

Part (a)

We start with the spin 1/2 Hamiltonian, and apply the mean field approximation:

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h_{\perp} \sum_i S_i^x \quad (13)$$

$$\approx -J \sum_{\langle ij \rangle} S_i^z \langle S_j^z \rangle + \langle S_i^z \rangle S_j^z - \langle S_i^z \rangle \langle S_j^z \rangle - h_{\perp} \sum_i S_i^x \quad (14)$$

$$= \sum_i (-zJm) S_i^z - h_{\perp} S_i^x + \text{const} \quad (15)$$

$$= - \sum_i \mathbf{h} \cdot \mathbf{S} \quad \text{with } \mathbf{h} = (h_{\perp}, 0, zJm) \quad (16)$$

where $m = \langle S_i^z \rangle$. Therefore, our mean field Hamiltonian can be written as a set of spins in an effect field $\mathbf{h}_{\text{eff}} = (h_{\perp}, 0, zJm)$

Part (b)

Now, we assume that we are at zero temperature and that each spin is aligned fully with the effect field \mathbf{h}_{eff} .

Therefore, we now that \mathbf{S} points in the direction $(h_{\perp}, 0, Jzm)$. We also know that $|\mathbf{S}| = S = 1/2$. Then, we can write

$$\mathbf{S}_i = \frac{1}{2} \frac{Jzm\hat{z} + h_{\perp}\hat{x}}{\sqrt{J^2z^2m^2 + h_{\perp}^2}} \quad (17)$$

Since this is the proper normalized spin-1/2 vector which points in the \mathbf{h}_{eff} direction.

Since all spins point in the same direction, the magnetization is just the z component of this spin vector

$$m = \langle S_i^z \rangle = \frac{1}{2} \frac{Jzm}{\sqrt{J^2z^2m^2 + h_{\perp}^2}} \quad (18)$$

$$\Rightarrow m^2 = \frac{J^2z^2m^2}{4(J^2z^2m^2 + h_{\perp}^2)} \quad (19)$$

$$0 = 4J^2z^2m^4 + (4h_{\perp}^2 - J^2z^2)m^2 \quad (20)$$

Part (c)

Therefore, for a given transverse field h_{\perp} , the magnetization is given by the solution to the equation

$$4J^2z^2m^4 + 4h_{\perp}^2m^2 - J^2z^2m^2 = 0 \quad (21)$$

Solving this gives

$$m^2(4J^2z^2m^2 + (4h_{\perp}^2 - J^2z^2)) = 0 \quad (22)$$

$$\Rightarrow m = 0 \quad \text{or} \quad m = \pm \sqrt{\frac{1}{4} - \frac{h_{\perp}^2}{J^2z^2}} \quad (23)$$

We see that for $\frac{h_{\perp}^2}{J^2z^2} > \frac{1}{4}$, the only solution to this equation is at $m = 0$. Therefore the critical field h_c occurs when $\frac{h_{\perp}^2}{J^2z^2} = \frac{1}{4}$ (i.e when $h_{\perp} = Jz/2$). When $h_{\perp} > h_c$, we have that $m(h_{\perp}) = 0$

Therefore

$$m(h_{\perp}) = \pm \sqrt{\frac{1}{4} - \frac{h_{\perp}^2}{J^2z^2}}, \quad \text{for } h_{\perp} < h_c \quad \text{where } h_c = \frac{Jz}{2} \quad (24)$$

Problem 3 - Low Temperature Magnetization of the Heisenberg Ferromagnet

$$\langle S_{\text{TOT}}^z \rangle / N = +S - \int_{\varepsilon=0}^{\infty} n_B(\varepsilon) d\varepsilon \quad (25)$$

$$(26)$$

The number of magnons at a given temperature T is given by

$$\sum_k n_k = \int d\varepsilon D(\varepsilon) \langle n(\varepsilon) \rangle \quad (27)$$

The factor of $D(\varepsilon)$ is the number of frequency modes k which exist at a given energy ω . For a given wavevector k , we have the number of states within a sphere of radius k is $N(k) = (2\pi)^{-3} (\frac{4}{3}\pi k^3)$. Then the number of states within a shell of thickness $d\varepsilon$ is:

$$D(\varepsilon) d\varepsilon = \frac{dN(k)}{dk} \frac{dk}{d\varepsilon} d\varepsilon = \frac{1}{(2\pi)^3} 4\pi k^2 \frac{dk}{d\varepsilon} d\varepsilon \quad (28)$$

$$(29)$$

Therefore

$$\langle S_{\text{TOT}}^z \rangle / N = S - \frac{4\pi}{(2\pi)^3} \int_{k=0}^{k_F} \frac{k^2}{e^{\beta ck^2} - 1} dk \quad (30)$$

$$\approx S - \frac{1}{2\pi^2} \int_0^{\infty} \frac{k^2}{e^{\beta ck^2} - 1} dk \quad (31)$$

$$(32)$$

where we make the approximation that the upper bound on the integral goes to infinity since $n_B(k) \rightarrow 0$ exponentially fast as k becomes large.

Now, let $x = \beta ck^2$ So that $dx = 2\beta ck dk$ and we can write the integral above as

$$\langle S_{\text{TOT}}^z \rangle / N = S - \frac{1}{2\pi^2} \int_0^{\infty} \frac{dx}{2\beta c} \frac{x}{\sqrt{\beta c}} \frac{1}{e^x - 1} \quad (33)$$

$$= S - \frac{1}{4\pi^2 c} (kT)^{3/2} \int_0^{\infty} \frac{x}{e^x - 1} dx \quad (34)$$

$$= S - \frac{1}{12} \frac{(kT)^{3/2}}{c} \quad (35)$$

where in the last line I used the fact that the integral over x is equal to $\pi^2/6$