Physics 123B: Homework 6

due March 8, 5pm in the box at the PSR

1. **Non-linear $M(H)$:** Use mean field theory to show that, for the Heisenberg ferromagnet, $M/N = c\,\text{sign}(H)|H|^x$, for small H, when $T = T_c$. Find the constant c and exponent x.

2. **Quantum transverse-field Ising model:** Consider the $S = 1/2$ Ising model in a transverse field, defined by

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h_\perp \sum_i S_i^x. \tag{1}$$

   Assume that the lattice consists of all identical sites, with z nearest-neighbors per site. Note that, unlike in the Heisenberg model, only the z components of spins couple between nearest-neighbor sites.

   (a) Apply the mean-field approximation to decouple the first term to obtain a set of spins in an effective field $\vec{h}_{\text{eff}} = (h^x_{\text{eff}}, 0, h^z_{\text{eff}})$. Find h^z_{eff} in terms of the Ising magnetization $m \equiv \langle S_i^z \rangle$.

   (b) At *zero temperature*, you can assume each spin is aligned fully by its effective field. Using this assumption, find the self-consistent equation for m.

   (c) Solve the above equation to find $m(h_\perp)$. What is the critical field $h_\perp^c$ above which $m = 0$? This is a *quantum critical point.*

3. **Low temperature magnetization of the Heisenberg ferromagnet:** In the quantum Heisenberg ferromagnet, the low energy excitations are ferromagnetic magnons, or spin waves, whose energy versus wavevector is quadratic at small k, $\epsilon(k) = ck^2$, with c proportional to J. Such magnons are bosons, whose number is described by the Bose-Einstein distribution function $n_B(\epsilon) = 1/(e^{\beta\epsilon} - 1)$. If the ground state is the ferromagnetic state with $S^z_{\text{TOT}}/N = +S$ (for spin S spins), each magnon carries spin $S^z = -1$, i.e. reduces the total spin by one quantum. By adding up the reduction from the modes at all k, calculate the low temperature magnetization. By converting the sum over k to an integral in the thermodynamic limit, show that

$$\langle S^z_{\text{TOT}} \rangle / N = S - A \left(\frac{kT}{c} \right)^{3/2}. \tag{2}$$

   This result is called the "Bloch $T^{3/2}$ law".