

# Superfluidity and Superconductivity

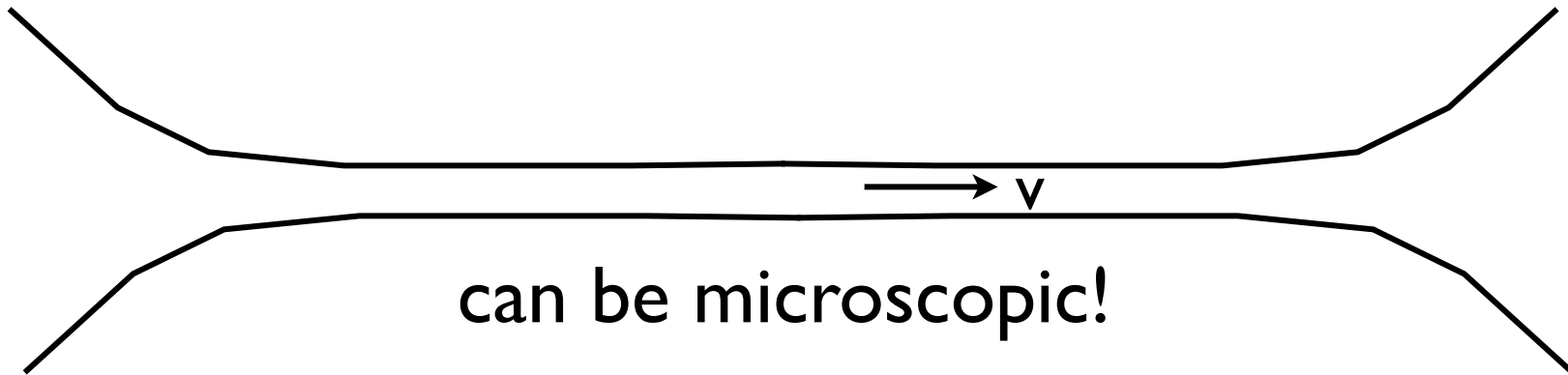
- These are related phenomena of flow without resistance, but in very different systems
- Superfluidity: flow of helium IV *atoms* in a liquid
- Superconductivity: flow of *electron charge* inside a solid
- History:
  - Superconductivity discovered in 1911 by Kamerlingh Onnes
  - Superfluidity discovered in 1938 by Kapitza
  - delay due to “critical temperature”:  $T_c$  of Hg is 4.2K,  $T_c$  of He IV is 2.17K, so superconductivity could be discovered by helium refrigeration

# Superfluidity

- We start with it, because it is conceptually simpler than superconductivity
- It is however, much more rare: only helium is superfluid (except in ultra-cold nanoK atomic traps), but many metals are superconducting
- Why?
  - Superflow is a quantum effect, and lighter particles are more quantum
  - Helium is (almost) the lightest atom, but it is still much heavier than an electron!

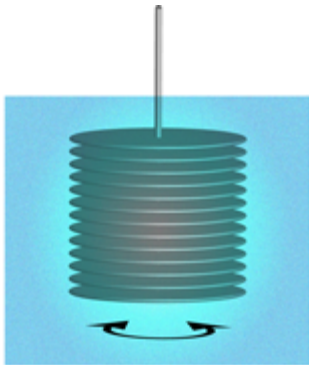
# Superfluid properties

- Frictionless flow: helium will flow through a narrow channel without friction (no pressure drop) up to a critical velocity



# Superfluid properties

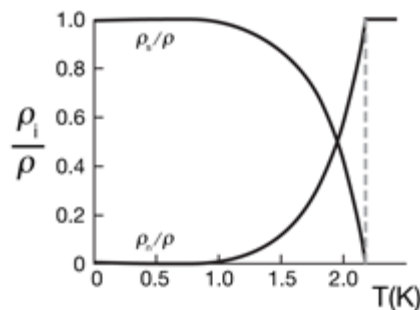
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Andronikashvili experiment:

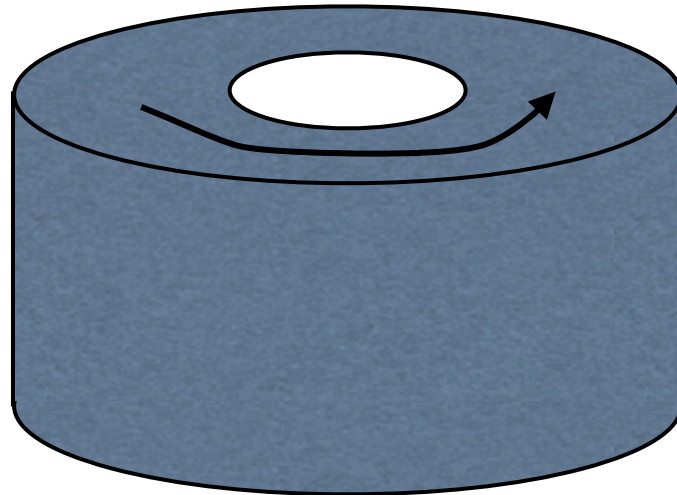
apparent mass of torsional oscillator drops below superfluid temperature

superfluid fraction



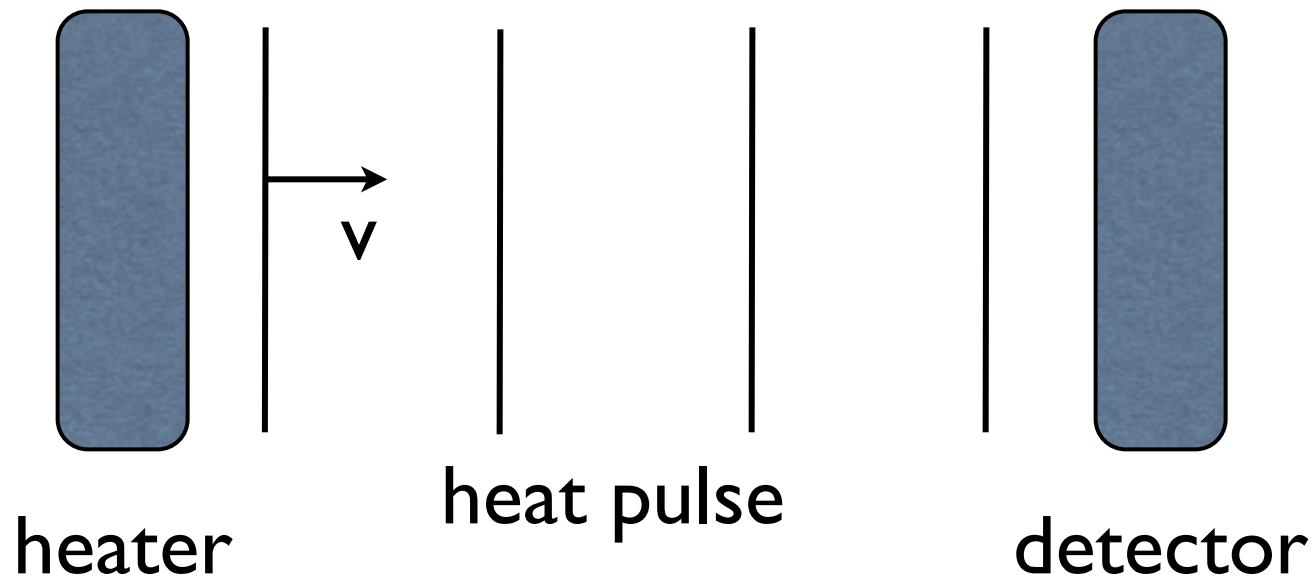
# Superfluid properties

- Persistent current
  - Superfluid in an annulus (ring) will flow “forever”



# Superfluid properties

- Second sound
  - Heat pulse in superfluid propagate ballistically like a wave, instead of diffusing



# Bose-Einstein Condensation

- Superfluidity is a manifestation of *macroscopic quantum coherence*
- Starting picture: BEC
- Unlike fermions, bosons *like* to occupy the same quantum state (explains why hydrogen is not superfluid. Also it is remarkable that He III - just a different *isotope* - behaves totally differently than He IV. One neutron in the nucleus matters!)
- Free (non-interacting) bosons are described by the Bose-Einstein distribution:

$$n_b(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}$$

average number of bosons  
in a state with energy  $\epsilon$

$$\beta = 1/(k_B T)$$

# Bose-Einstein Condensation

- Free bosons

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m} \quad \mu \leq 0$$

- Total number

$$N = \sum_k \frac{1}{e^{\beta(\frac{\hbar^2 k^2}{2m} - \mu)} - 1} \approx V \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\beta(\frac{\hbar^2 k^2}{2m} - \mu)} - 1}$$

$$\rho = \frac{1}{2\pi^2} \int dk \frac{k^2}{e^{\beta(\frac{\hbar^2 k^2}{2m} - \mu)} - 1} = \frac{1}{2\pi^2 \hbar^3} \left(\frac{m}{\beta}\right)^{3/2} I(\bar{\mu})$$

$$I(\bar{\mu}) = \int_0^\infty dk \frac{k^2}{e^{\frac{k^2}{2} - \bar{\mu}} - 1} \leq I(0) = \sqrt{\frac{\pi}{2}} \zeta(3/2) \quad \bar{\mu} = \beta\mu$$

# Bose-Einstein Condensation

- Minimum density

$$\rho \leq \frac{1}{2\pi^3 \hbar^3} \left( \frac{m}{\beta} \right)^{3/2} \sqrt{\frac{\pi}{2}} \zeta(3/2)$$

- At high temperature, i.e. small  $\beta$ , this is always satisfied. But below some  $T$  it is not. Defines

$$k_B T_c = \frac{2\pi \hbar^2}{m} \left( \frac{\rho}{\zeta(3/2)} \right)^{2/3} \approx 3.31 \frac{\hbar^2 \rho^{2/3}}{m}$$

- What happens for  $T < T_c$ ?

# BEC

- Below  $T_c$ , the chemical potential gets “almost” to zero. Then  $n(k=0)$  becomes *macroscopic*
- So for  $T < T_c$ , a macroscopic fraction of bosons occupies *one* quantum state
- This one state extends over the whole system, and basically corresponds to the superfluid
- Caveat: BEC as described really applies to non-interacting bosons. He IV atoms are very strongly interacting. But still the BEC starting point is qualitatively good. And we mostly just need the idea.

# Landau-type theory

- Idea: the key property of the superfluid is a macroscopically occupied quantum state. We describe it with a *macroscopic wavefunction*

$$\psi(r) = \sqrt{n_s(r)} e^{i\theta(r)}$$

- Here  $n_s$  is the *superfluid density* and  $\theta$  is the phase of the superfluid
- In equilibrium  $n_s$  is constant and so is  $\theta$ . But there may be very stable non-equilibrium states where they vary : states with flow!

# Free energy

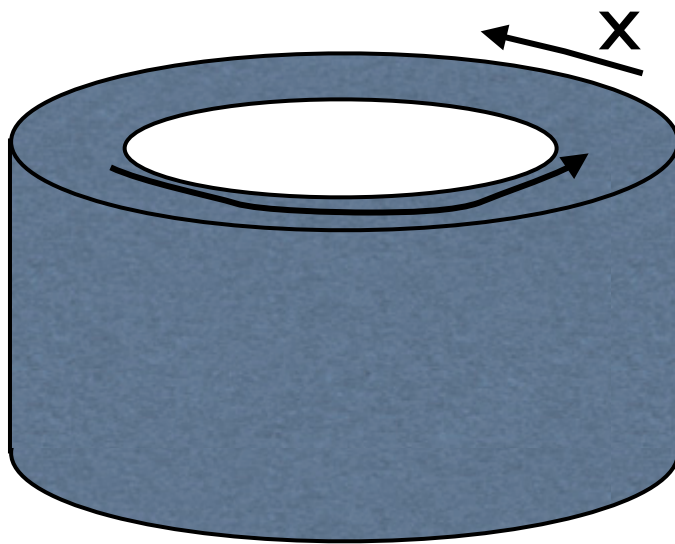
- We assume that the free energy of the system (which tends to its minimum in equilibrium) can be written in terms of  $n_s$  and  $\theta$ :

$$F = \int d^3r \left[ \frac{c}{2} (n_s - \bar{n}_s)^2 + \frac{\hbar^2 n_s}{2m} (\nabla \theta)^2 \right]$$

- Understand second term by noting that  $\theta = \mathbf{k} \cdot \mathbf{r}$  is like putting all the “condensed” atoms into a state with momentum  $\hbar \mathbf{k}$
- Superfluid velocity  $\mathbf{v}_s = \frac{\hbar}{m} \nabla \theta$

# Persistent current

- Obvious free energy is minimized by  $v_s=0$ .  
But suppose we set up a superflow?



narrow cylinder of  
circumference  $L$

$$v_s^x = \frac{\hbar}{m} \partial_x \theta \quad \longrightarrow \quad \theta = \frac{mv_s}{\hbar} x$$

single valued wavefunction

$$\theta(L) - \theta(0) = \frac{mv_s}{\hbar} L = 2\pi N$$

“quantized circulation”

$$v_s = \frac{h}{mL} N$$

it is very hard for circulation to decay  
because it cannot change continuously!

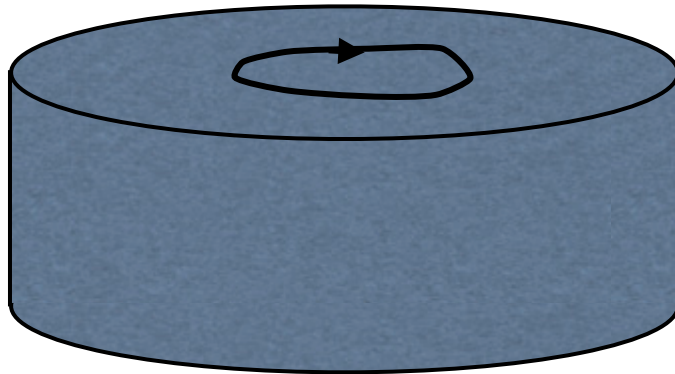
# Vortices

- What happens if you rotate a bucket (not an annulus) above  $T_c$ , then cool it, then stop the bucket?
  - above  $T_c$  fluid has friction so will rotate
  - below  $T_c$ , has to keep rotating by conservation of angular momentum
- But if there is no center of the bucket, then there is a problem

# Vortices

- Stoke's theorem

$$\oint \nabla \theta \cdot dr = \int \nabla \times \nabla \theta \cdot dA = 0$$

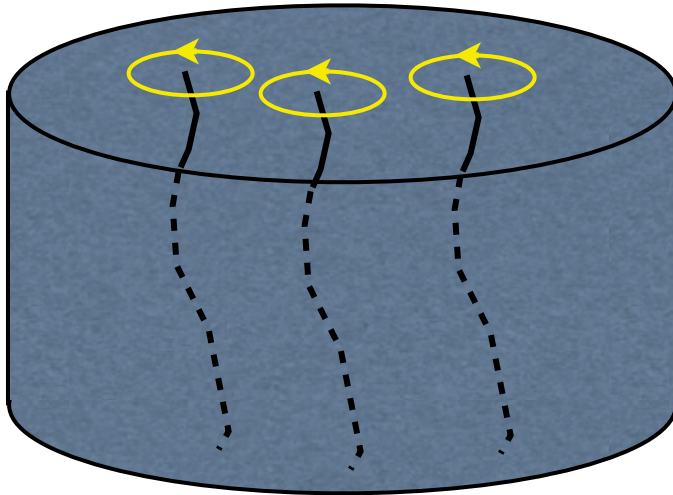


no circulation?

- In fact, circulation exists because *vortices* form. These are places in the liquid where  $n_s \rightarrow 0$ , and so  $\theta$  is not well-defined!

# Vortices

$$\theta \rightarrow \theta + 2\pi$$



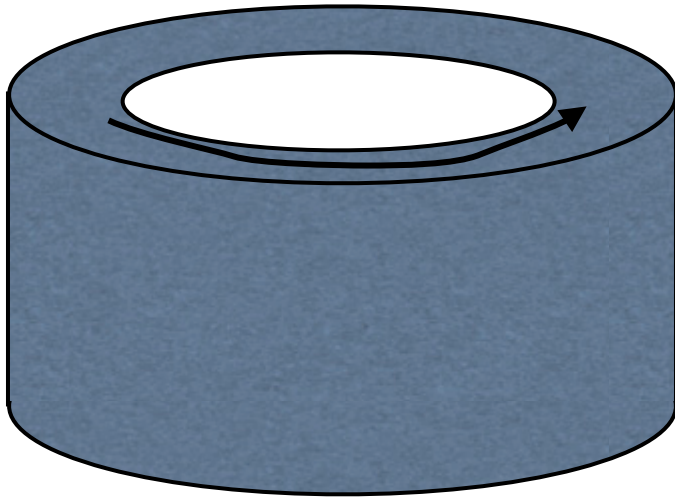
an array of vortices  
simulates rigid rotation of  
the fluid

$$\oint \nabla\theta \cdot dr = 2\pi \text{ around a single vortex}$$

- Vortex must escape the system - moving into the wall - to lower the circulation
- They will escape, because the vortex costs free energy - especially the core where  $n_s = 0$ , but also the  $v_s \neq 0$  outside

# Persistent current

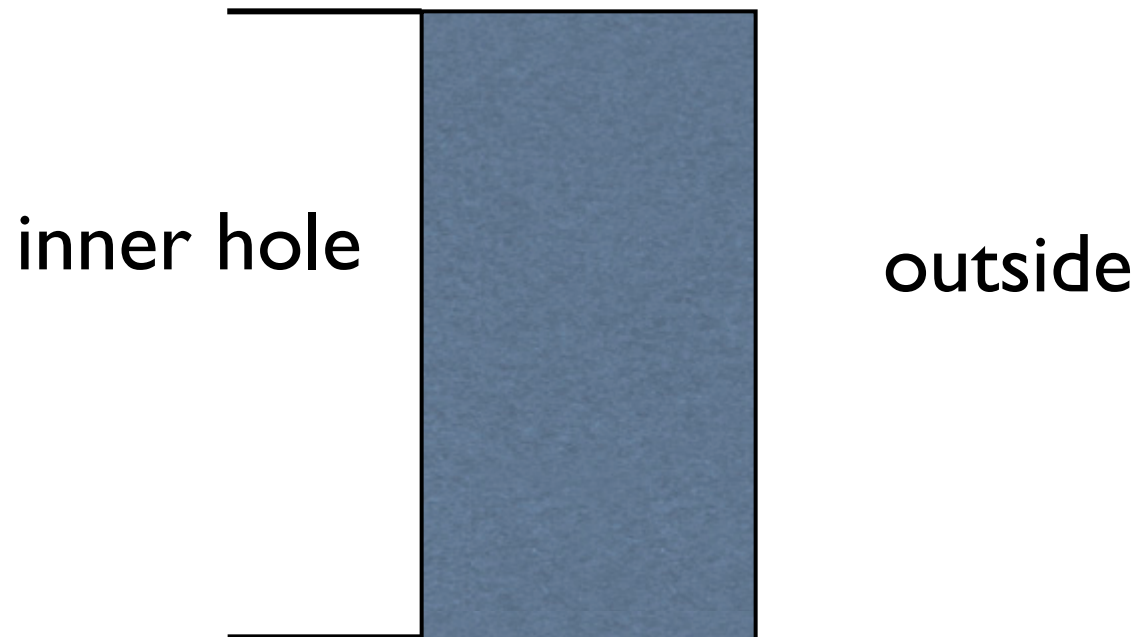
- In an annulus, one has a “giant vortex” in the hole: no core energy



- To decay, the circulation must escape *one vortex at a time*

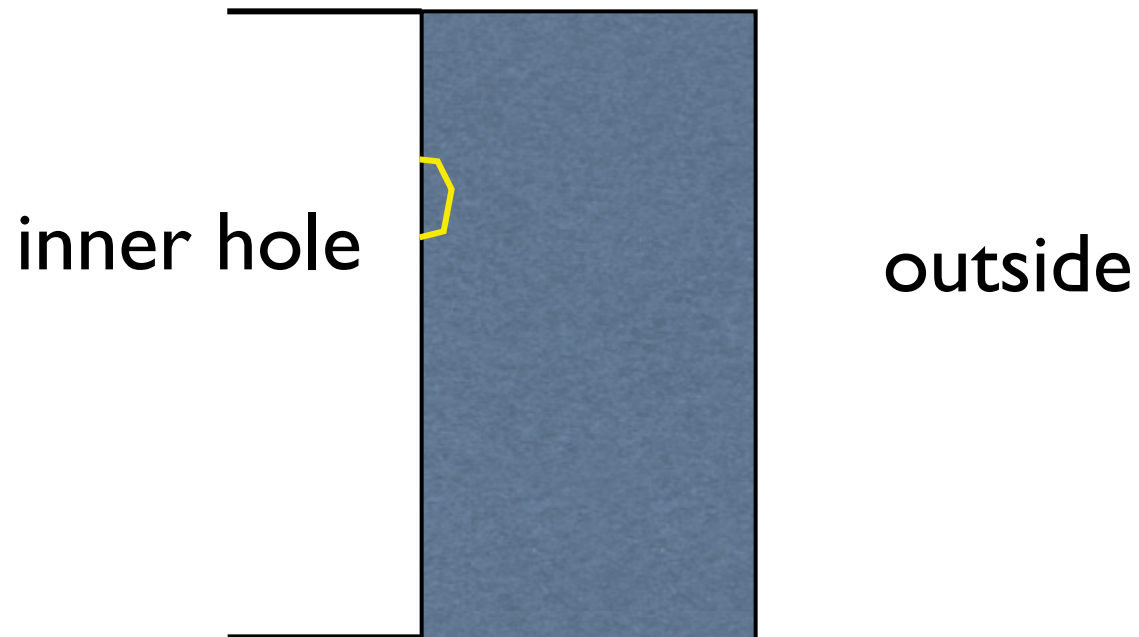
# Vortex escape

- radial cut of annulus



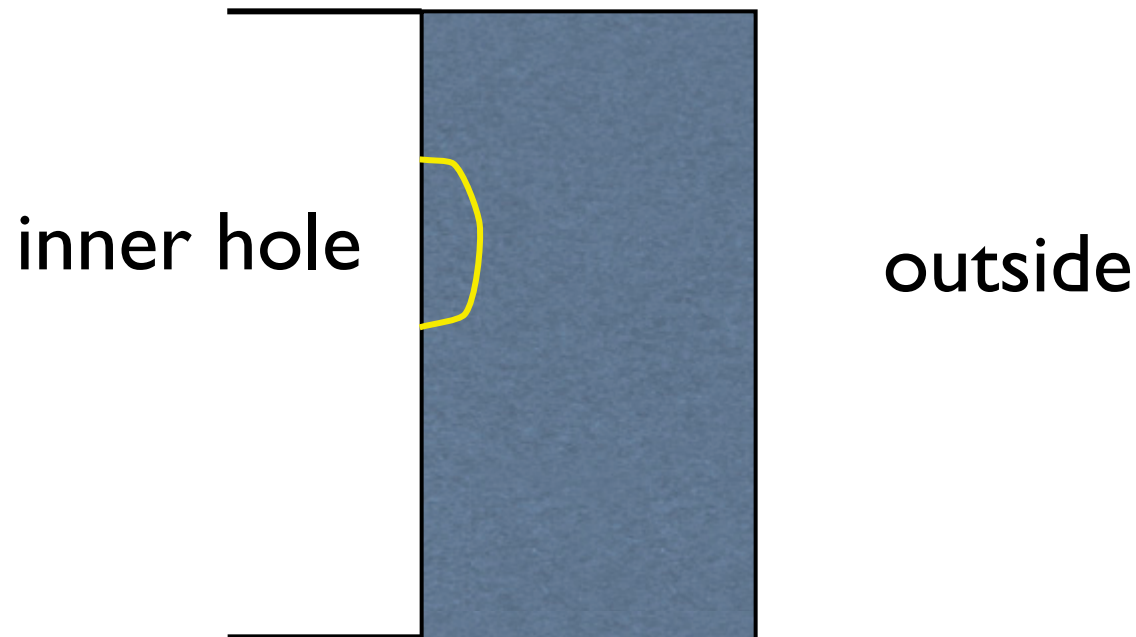
# Vortex escape

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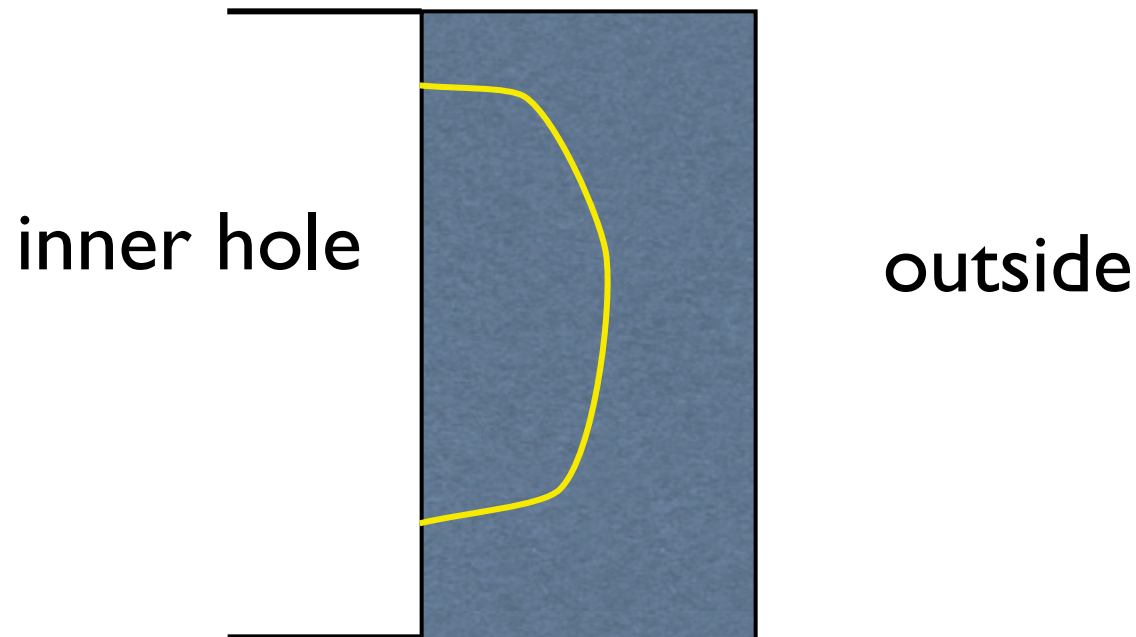
# Vortex escape

- radial cut of annulus



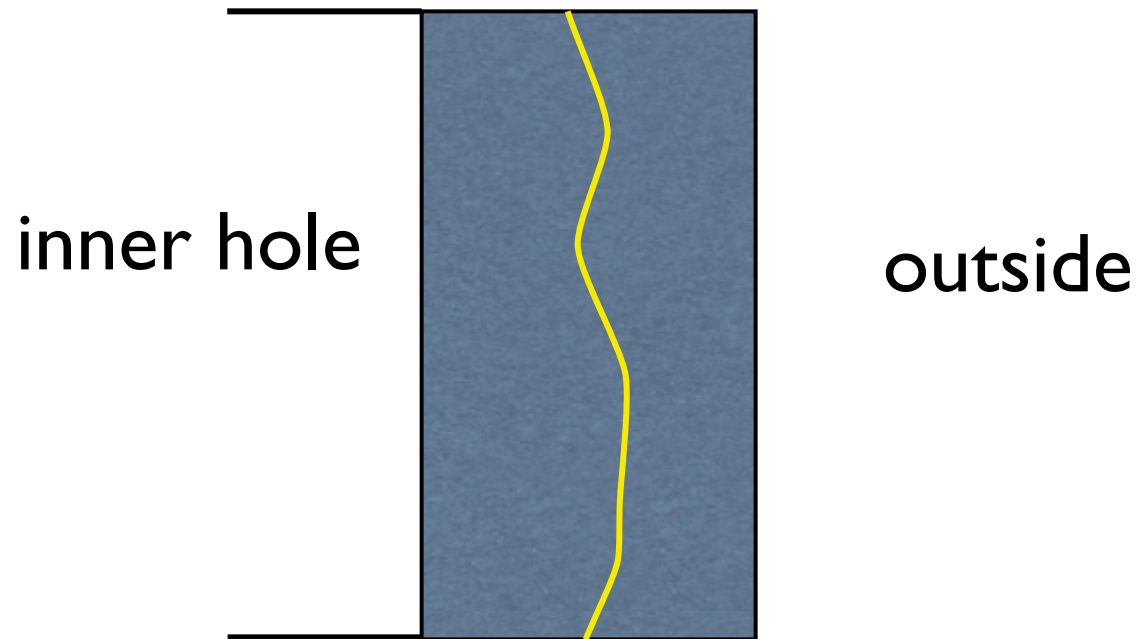
# Vortex escape

- radial cut of annulus



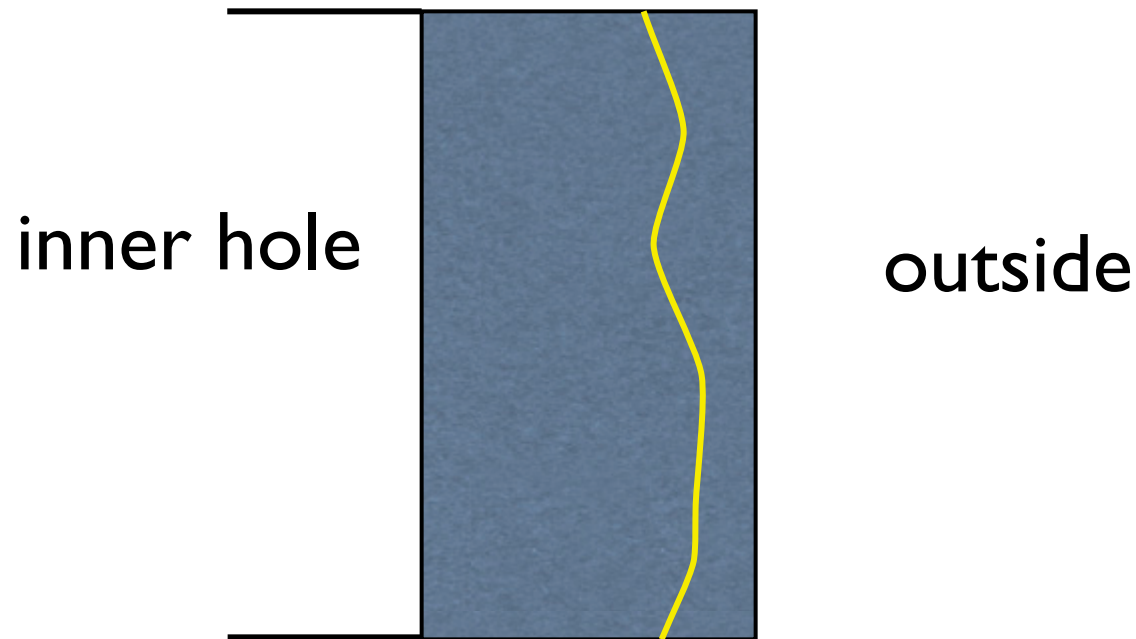
# Vortex escape

- radial cut of annulus



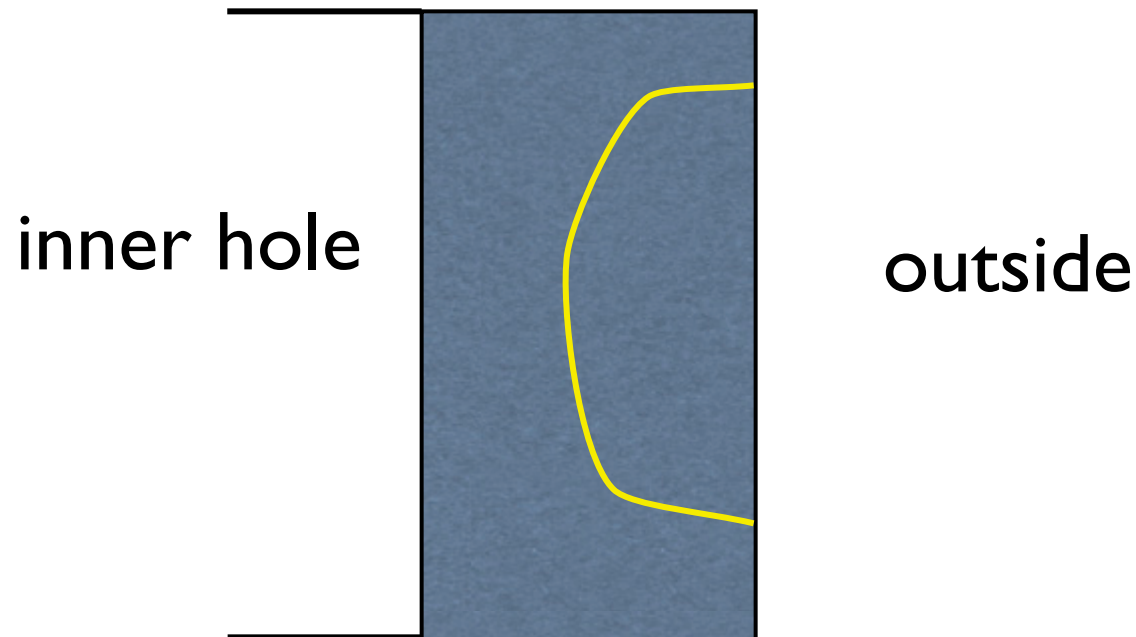
# Vortex escape

- radial cut of annulus



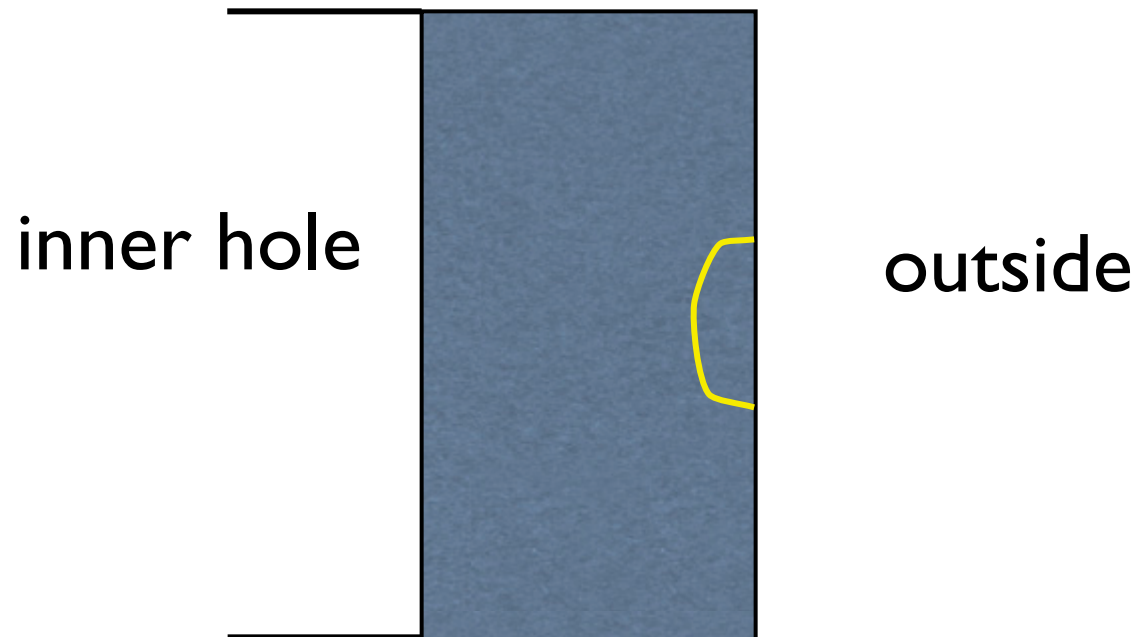
# Vortex escape

- radial cut of annulus



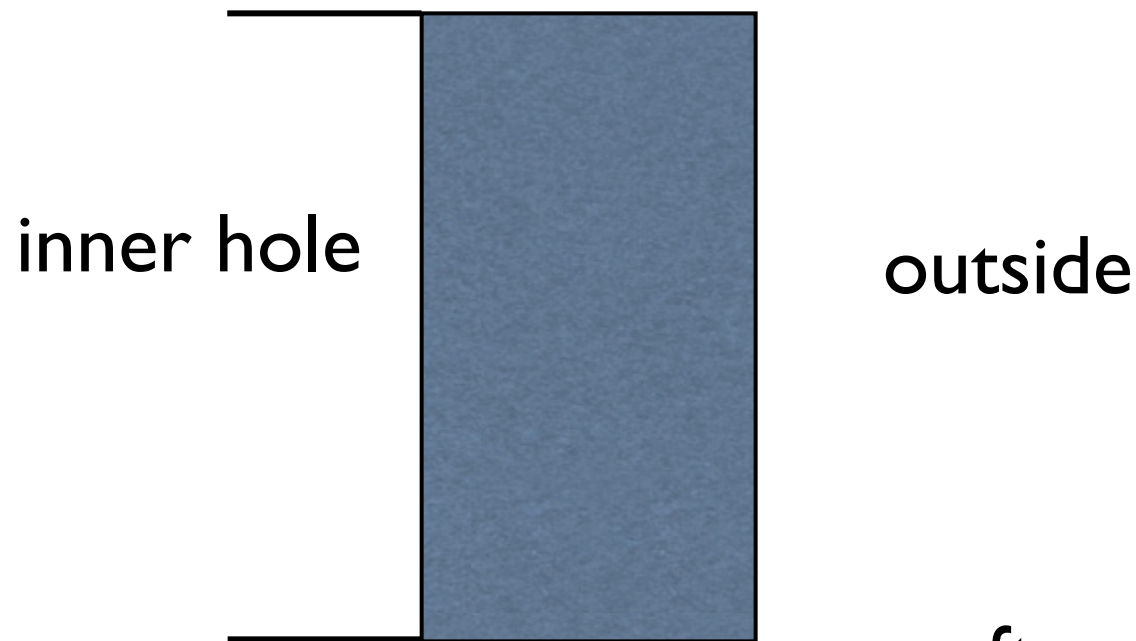
# Vortex escape

- radial cut of annulus



# Vortex escape

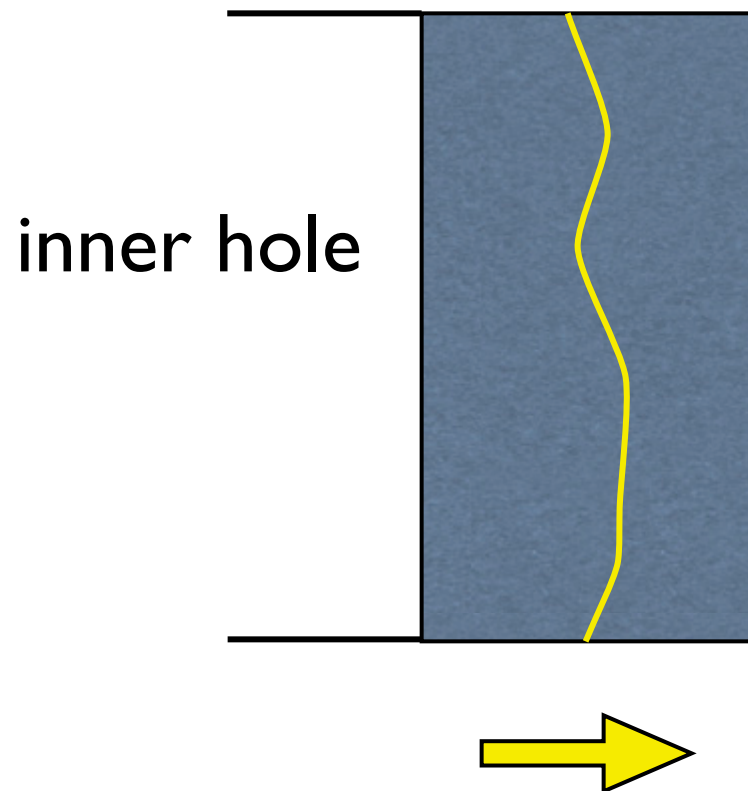
- radial cut of annulus



after this process, one  
quantum of circulation has  
been removed

# Vortex escape

- radial cut of annulus



the intermediate state contains a long vortex line, which costs free energy. This free energy barrier requires thermal activation to overcome

after this process, one quantum of circulation has been removed

# Vortex free energy

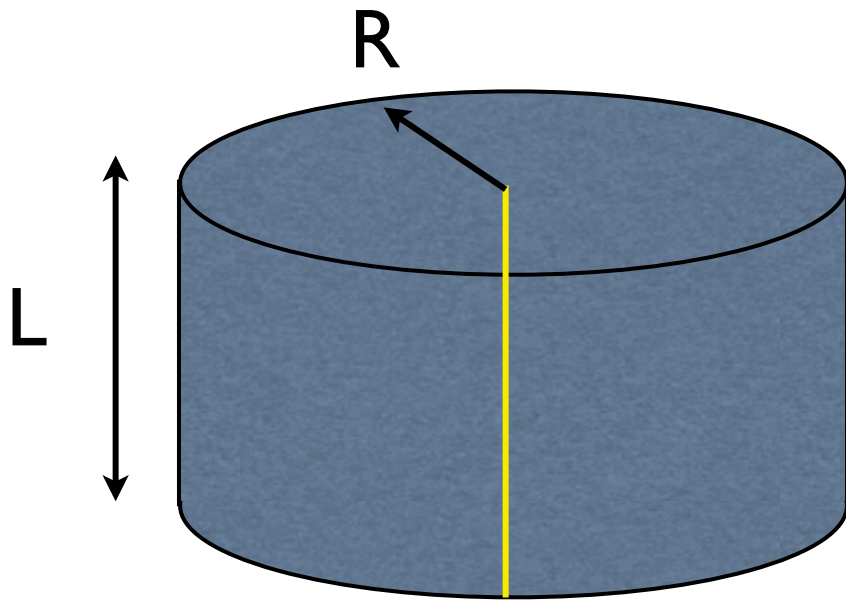
- Crude estimate

$$F = \int d^3r \frac{\hbar^2 n_s}{2m} (\nabla\theta)^2 \quad |\nabla\theta| = 1/r$$

$$F \approx L \int_a^R dr 2\pi r \frac{\hbar^2 n_s}{2m} \left(\frac{1}{r}\right)^2$$

$$\approx \frac{\pi \hbar^2 n_s L}{m} \ln(R/a)$$

even neglecting the log,  
this is of order  $0.1 \text{ K}/\text{\AA}$



rate of thermal activation  $\sim \exp(-F/k_B T)$

since  $T < 2.2\text{K}$ , this is negligible even for sub-micron dimensions

# Second sound

- How does superfluid helium carry heat so quickly?
- Usually, heat diffuses if there is no fluid flow
- But superfluid helium behaves like *two fluids*: a normal and superfluid part
- Second sound = opposite flow of superfluid and normal fluids: no net mass flow but net heat flow
- This is because the normal fluid convects heat, but the superfluid has *zero entropy* because it is a *single quantum state*!