11 THE SPECIAL THEORY OF RELATIVITY

11.1 The Need for a New Mode of Thought

In some ways the structure of physics resembles a mansion whose outward form is apparent to the casual visitor but whose inner life —the customs and rituals which give a special outlook and kinship to its occupants—require time and effort to comprehend. Indeed, initiation into this special knowledge is the goal of our present endeavor. In the first ten chapters we introduced and applied the fundamental laws of classical mechanics; hopefully you now feel familiar with these laws and have come to appreciate their beauty, their essential simplicity, and their power.

Unfortunately, in order to present dynamics in a concise and tidy form, we have generally sidestepped discussion of how physics actually grew. In Chaps. 11 through 14 we are going to discuss one of the great achievements of modern physics, the special theory of relativity. Rather than present the theory as a completed structure—a simple set of postulates with the rules for their application—we shall depart from our previous style and look into the background of the theory and its rationale.

If the structure of physics is a mansion, it is a mansion of ancient origin. It is founded on the remains of prehistoric hovels where man first kept track of the moon and tried to understand the simple patterns of nature. Traces of antiquity lie hidden in the site: Phoenician and Egyptian, Babylonian, and, of course, Greek. Compass and straightedge lie scattered among lodestone and amber, artifacts of astrologer and alchemist. The mansion is built on the debris of false starts and painful struggles to understand nature honestly. None of this is visible, however, and we take the present structure much for granted. The outer shell was built in the seventeenth century by Kepler, Galileo, Newton, and others, such as Huygens, Hooke, Leibniz, Bernoulli, and Boyle. The major architects have one characteristic in common: while extending the external dimensions of the mansion by applying physics to new areas, they also deepened its foundations by advancing our knowledge of the fundamental laws. The greatest of these figures is Newton, who revealed the laws of dynamics and of gravity, cornerstones of modern science. At the same time he vigorously applied physics to the natural world. Newton executed meticulous experiments in heat flow, optics, and the motion of bodies under viscous forces; he investigated the shape of the moon, the tides along the coast of England, and how to build bridges.

The momentum generated by Newton's discoveries gave physics an impetus which is still very much with us. The eighteenth and nineteenth centuries saw a flowering of science as physicists such as Euler, Lagrange, Laplace, Faraday, and Maxwell extended our knowledge of the physical world. However, their efforts were directed at upward extension of the mansion; Newton's account of the fundamental laws of physics was so overwhelming, and so successful, that not until the last quarter of the nineteenth century was there a serious attempt to investigate the foundations.

It was the German physicist Ernst Mach who first successfully challenged newtonian thought. Although Mach's work left newtonian physics more or less intact, his thinking was crucial in the revolution shortly to come. In 1883 Mach published his text "The Science of Mechanics," which incorporated a critique of newtonian physics, the first incisive criticism of Newton's theory of dynamics. In addition to presenting a lucid account of newtonian mechanics, the text incorporates several significant contributions to the fundamentals of mechanics. Mach clarified newtonian dynamics by carefully analyzing Newton's explanation of the dynamical laws, taking care to distinguish between definitions, derived results, and statements of physical law. Mach's approach is now widely accepted; our discussion of Newton's laws in Chap. 2 is very much in Mach's spirit.

"The Science of Mechanics" raised the question of the distinction between absolute and relative motion. Mach pointed out Newton's ambivalence on this subject, although he went on to show that the question was irrelevant to the application of newtonian dynamics. In the process he dwelt on the problem of inertia and enunciated the principle that now bears his name: inertia is not an intrinsic property of matter or space but depends on the existence of all matter in the universe. We encountered Mach's principle in our discussion of fictitious forces in Chap. 8, but we shall not dwell on it here for it turns out that the problem of inertia was not the crucial difficulty with newtonian mechanics.

The fundamental weakness in newtonian dynamics, as Mach pointed out, centers on Newton's conception of space and time. In a preface to his dynamical theory, Newton avowed that he would forgo abstract speculation and deal only with observable facts. Although such a point of view is now commonplace, at the time it represented a tremendous intellectual leap. Before Newton, the business of natural philosophy was to explain the reasons for things, to find a rational account for the workings of nature, rather than to describe natural phenomena quantitatively. Newton essentially reversed the priorities. Against the criticism that his theory of universal gravitation merely described gravity without accounting for its origin, Newton replied "I do not make hypotheses."

Unfortunately, Newton was not completely faithful to his resolve to avoid abstract speculation and to deal only with demonstrable facts. In particular, consider the following description of time that appears in the "Principia." (The excerpt is condensed.)

> Absolute, true and mathematical time, of itself and by its own true nature, flows uniformly on, without regard to anything external.
>
> Relative, apparent and common time is some sensible and external measure of absolute time estimated by the motions of bodies, whether accurate or inequable, and is commonly employed in place of true time; as an hour, a day, a month, a year.

Mach comments that "it would appear as though Newton in the remarks cited here still stood under the influence of medieval philosophy, as though he had grown unfaithful to his resolve to investigate only actual facts." Mach goes on to point out that since time is necessarily measured by the repetitive motion of some physical system, for instance the pendulum of a clock or the revolution of the earth about the sun, then the properties of time must be connected with the laws which describe the motions of physical systems. Simply put, Newton's idea of time without clocks is metaphysical; to understand the properties of time we must observe the properties of clocks. A simple idea? Yes, indeed, were it not for the fact that the idea of absolute time is so natural that the eventual consequences of Mach's position, the relativistic description of time, still come as something of a shock to the student of science.

There are similar weaknesses in the newtonian view of space. Mach argued that since position in space is determined with measuring rods, the properties of space can be understood only by investigating the properties of meter sticks. We must look to nature to understand space, not to platonic ideals.

Mach's special contribution was to examine the most elemental aspects of newtonian thought, to look critically at matters which seem too simple to discuss, and to insist that we turn to experience to understand the properties of nature rather than to rely on abstractions of the mind. From this point of view, Newton's assumptions about space and time must be regarded merely as postulates. Classical mechanics follows from these postulates, but other assumptions are possible and from them different laws of dynamics could follow.

Mach's critique had little immediate effect, but its influence was eventually profound. In particular, the youthful Einstein, while a student at the Polytechnic Institute in Zurich in the period 1897–1900, was much attracted by Mach's ideas on the foundations of newtonian physics and by Mach's insistence that physical concepts be defined in terms of observables. However, the immediate cause for the overthrow of newtonian physics was not Mach's criticisms of newtonian thought. The difficulties lay with Maxwell's electromagnetic theory, the crowning achievement of classical physics. Traditionally, the problem is presented in terms of a single crucial experiment that decisively condemned classical physics, the Michelson-Morley experiment, and most treatments of special relativity take this experiment as the point of departure. We shall follow this tradition, but we should point out that history is not that simple. In the first place, Albert A. Michelson, who conceived and executed the experiment, never regarded it as crucial. Michelson viewed the experiment as a flop for not giving the expected result, a view he maintained long after its full significance became known. Furthermore, it now appears that the Michelson-Morley experiment played little, if any, role in Einstein's thinking. In fact, there is good reason to believe that Einstein knew nothing of the experiment until after he had published his theory of relativity in 1906. Nevertheless, the Michelson-Morley experiment so clearly dramatizes the essential dilemma of electromagnetic theory that we shall bow to tradition and take it as our starting point.

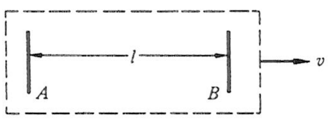
11.2 The Michelson-Morley Experiment

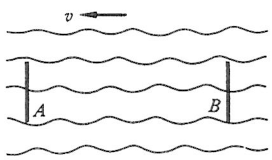
The problem to which Michelson devoted himself was that of determining the effect of the earth's motion on the velocity of light. Briefly, Maxwell's electromagnetic theory (1861) predicted that electromagnetic disturbances in empty space would propagate at 3×10^8 m/s—the speed of light. The simplest disturbance is a periodic wave, and the evidence was overwhelming that light consisted of electromagnetic waves. However, there were conceptual difficulties.

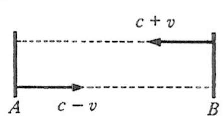
The only waves previously known to physics were mechanical waves propagating in solids, liquids, and gases. A sound wave in air, for example, consists of alternate regions of higher and lower pressure propagating with a speed of 330 m/s, somewhat less than the speed of molecular motion. The speed of mechanical waves in metals is higher, typically 5,000 m/s, and increases with the strength of the "spring forces" between neighboring atoms.

Electromagnetic wave propagation seemed to be a very different sort of thing. The ether, the medium which supposedly supported the electromagnetic disturbance, had to be immensely rigid to give a speed of 3×10^8 m/s. At the same time it had to be insubstantial enough not to interfere with the motion of the planets. Maxwell's theory itself made no essential reference to the ether, but Maxwell and his contemporaries were unable to accept the idea of waves propagating in empty space.

The speed of a sound wave v_s depends on the properties of the medium. If we observe a sound wave from a coordinate system moving relative to the medium, the speed of sound will appear to be greater or less than v_s, depending on whether we move in the direction of propagation or against it. Similarly, Maxwell pointed out that the speed of the earth as it circled the sun, 3×10^4 m/s, should change the apparent speed of light.







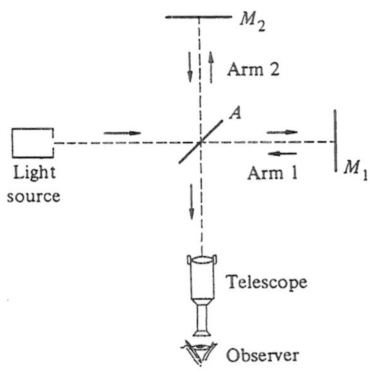
Suppose that light makes a round trip ABA between two points A and B separated by distance l. The apparatus is moving through the ether to the right, as shown in the upper drawing. Relative to the apparatus, the ether is moving to the left, as shown in the second drawing. The velocity of light relative to the apparatus is $c + v$ to the left, and $c - v$ to the right.

The transit time from A to B is $t_1 = l/(c - v)$, and from B to A it is $t_2 = l/(c + v)$. If the apparatus were at rest, t_1 and t_2 would have the value $t_0 = l/c$. The effect of the earth's motion is to delay the return of the light signal by

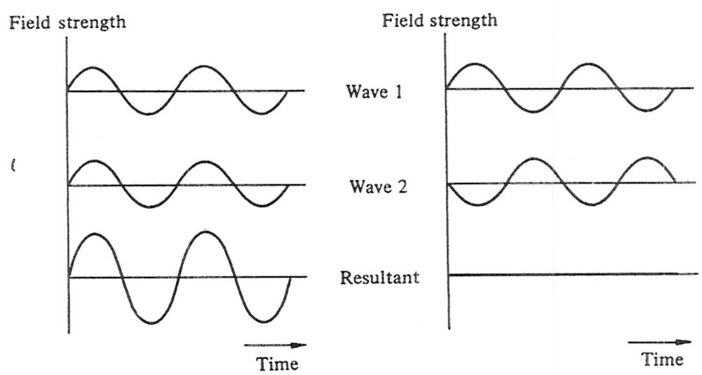
$$\begin{aligned}\Delta t &= t_1 + t_2 - 2t_0\\ &= \frac{l}{c - v} + \frac{l}{c + v} - 2\frac{l}{c}\\ &= \frac{l}{c}\left(\frac{1}{1 - v/c} + \frac{1}{1 + v/c} - 2\right)\\ &= 2\frac{l}{c}\left(\frac{1}{1 - v^2/c^2} - 1\right)\\ &\approx 2\frac{l}{c}\frac{v^2}{c^2}.\end{aligned}$$

For the earth in orbit $v/c = 10^{-4}$, and if we take l to be typical of a laboratory apparatus, $l = 1$ m, then $\Delta t = 2 \times 1/(3 \times 10^8) \times$

$10^{-8} \approx 7 \times 10^{-17}$ s, an interval much too small to be measured directly. Fortunately, Michelson was not discouraged. In 1881 he came up with the following solution.



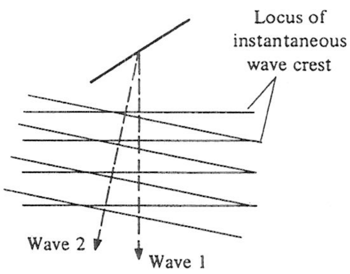
Rather than measure the time of transit of one light beam, Michelson observed the *difference* between the transit times of two beams. His device is sketched at the left. The light from the source is split into two beams by a thinly silvered mirror, A. Half the light passes through mirror A to mirror M_1, where it is reflected back to mirror A and then to the observer. The other half of the light from the source is diverted up the second arm and strikes mirror M_2, which reflects it to the observer. If the two arms are identical, the light waves recombine at mirror A just as if they had never separated: the observer sees an illuminated field of view. The situation is drastically altered if either beam suffers a delay. Suppose, for instance, that beam 1 is delayed by exactly one-half cycle of oscillation. The waves arrive in opposite phase and exactly cancel each other: the observer's field is dark.

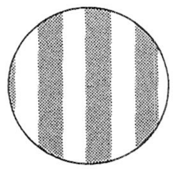


In phase
(*a*)

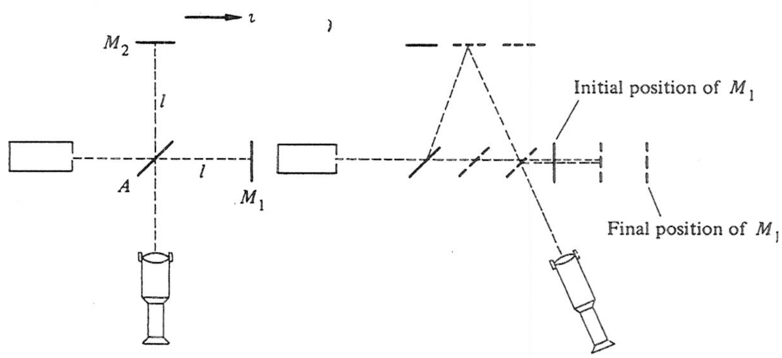
Out of phase
(*b*)

The two cases are shown in the sketches above. The vertical displacement corresponds to the strength of the electric field of light at the observer's eye. The fields of the two beams add vectorially. For visible light the period of the wave is typically 10^{-15} s, too fast for our eyes to follow. Rather, our eyes respond to the average power of the wave which is proportional to the square of the resultant field. Thus, beams in phase, sketch (*a*), give steady bright illumination, and beams out of phase, sketch (*b*), give darkness.





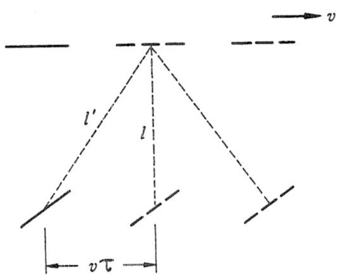
Usually one of the mirrors is slightly tilted. This produces a gradual time delay across the returning wavefront, as shown in the first sketch, and the two interfering waves go in and out of phase across the field of view. The observer sees alternate light and dark bands, as in the second sketch. If the length of either arm is changed, the fringe pattern shifts; a change in path of one wavelength shifts the pattern by one fringe. Since the light traverses each arm twice, once in each direction, a change in the length of either arm by one-half wavelength produces a shift of one fringe. With care it is possible to measure a small fraction of a fringe shift; one can readily observe a path change of one-hundredth wavelength, approximately 10^{-8} m. (Michelson also used his interferometer to measure the length of the standard meter bar; he essentially created the field of high precision measurement.)



Suppose that the interferometer is oriented so that one axis lies along the direction of motion of the earth, as shown. The time for the wave to travel from mirror A to mirror M_1 and back is

$$\begin{aligned} T_1 &= \frac{l}{c-v} + \frac{l}{c+v} \\ &= \frac{l}{c}\left(\frac{1}{1-v/c} + \frac{1}{1+v/c}\right) \\ &= \frac{2l}{c}\left(\frac{1}{1-v^2/c^2}\right) \approx \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right), \end{aligned}$$

where l is the length of the arm. There is also a time delay along arm 2, but this is a trifle more subtle to calculate. (Michelson overlooked it in the first report of his experiment in 1882.) For



the beam to return to its initial point on the thinly silvered mirror, it must traverse the angular path shown at left. Let τ be the time it takes the wavefront to go from mirror A to mirror M_2. The distance actually traversed is $l' = (l^2 + v^2\tau^2)^{\frac{1}{2}}$ and, since $l' = c\tau$ we have

$$\tau = \frac{(l^2 + v^2\tau^2)^{\frac{1}{2}}}{c}$$

or

$$\tau^2 = \frac{l^2}{c^2} + \frac{v^2}{c^2}\tau^2.$$

It follows that

$$\tau = \frac{l}{c}\frac{1}{\sqrt{1 - v^2/c^2}}.$$

The time for the wave to travel from mirror A to mirror M_2 and back is

$$\begin{aligned} T_2 &= 2\tau \\ &= 2\frac{l}{c}\frac{1}{\sqrt{1 - v^2/c^2}} \\ &\approx 2\frac{l}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right). \end{aligned}$$

The difference between the travel times of the beams is

$$\begin{aligned} \Delta T &= T_1 - T_2 \\ &= \frac{l}{c}\frac{v^2}{c^2}. \end{aligned}$$

The delay ΔT shifts the fringe pattern from where it would be if the earth were at rest. However, there is a major problem: the fringe scale has no "zero," since the arms cannot be made identical in length to the needed accuracy. Michelson hit upon the idea of watching the fringes as the apparatus is rotated by 90°. The rotation effectively interchanges arms 1 and 2. The change in the delay between the two positions is $2\Delta T$, and the corresponding fringe shift is readily calculated. If λ is the wavelength of the illuminating light, a time delay of λ/c will shift the pattern by one fringe. Thus, the time delay $2\Delta T$ will shift the pattern N fringes, where

$$N = \frac{2\Delta T}{(\lambda/c)} = \frac{2l}{\lambda}\frac{v^2}{c^2}.$$

If the arms have unequal lengths, l_1 and l_2, this result still holds, provided that we replace $2l$ by $l_1 + l_2$.

In Michelson's first apparatus, the arm length was 1.2 m, or, as he put it, 2×10^6 wavelengths of yellow (sodium) light. Since $v/c = 10^{-4}$, we expect

$$N = 2(2 \times 10^6)(10^{-4})^2 = 0.04.$$

Although this is not a large shift, Michelson had adequate resolution to see it. To his disappointment, he found no measurable shift in the fringe pattern. A much more refined experiment, executed with E. W. Morley, in 1887, used multiple reflections to increase the expected shift to 0.4 fringe. Although a shift as small as 0.01 fringe could have been detected, no effect was seen. The experiment has been repeated many times since, but always with negative results. It appears that we are unable to detect our motion through the ether.

11.3 The Postulates of Special Relativity

The elusive nature of the ether presented physics with a troublesome enigma. Maxwell attempted to devise a mechanical model of the ether, but as he continued to develop his theory of light, the ether played a less and less important role, until finally it was altogether absent. The ether vanished like the Cheshire Cat, leaving only a smile behind. After the Michelson-Morley experiment, even the smile had vanished. Numerous attempts to explain the null results of the Michelson-Morley experiment introduced such complexity as to threaten the foundations of electromagnetic theory. The most successful attempt was the hypothesis suggested independently by FitzGerald and by Lorentz that motion of the earth through the ether caused a shortening of one arm of the Michelson interferometer by exactly the amount required to eliminate the fringe shift. However, their speculations were based on an assumed model of atomic forces, and even though they arrived at some of the formulas shortly to be obtained by Einstein, their reasoning was far less general. Other theories which involved such artifacts as drag of the ether by the earth were even less productive.

The Universal Velocity

It is an indication of Einstein's genius that the troublesome problem of the ether pointed the way not to complexity and elaboration but to a simplification that unified the basic concepts of physics. Einstein viewed the difficulty with the ether not as stemming from a fault of electromagnetic theory but as arising from an error in basic dynamical principles. He argued that since the velocity of light predicted by electromagnetic theory, c, involves no reference to a medium, c must be a universal constant, the same for all observers. Thus, if we measure the speed of light from a source, the result will always be c, independent of our motion. This is in marked contrast to the case of sound waves, for example, where the observed speed depends on motion of the observer with respect to the medium. The ideas of a universal velocity was indeed a bold hypothesis, contrary to all previous experience and, for many of Einstein's contemporaries, defying common sense. But common sense is often a poor guide. Einstein once remarked that common sense consists of all the prejudices one learns before the age of eighteen.

The Principle of Relativity

The special theory of relativity involves one additional postulate—the assertion that the laws of physics have the same form with respect to all inertial systems. This principle, known as the principle of relativity, was not novel; Galileo is credited with first pointing out that there is no way to determine whether one is moving uniformly or is at rest, and Newton, although troubled by this point, gave it a rigorous expression in his dynamical laws in which acceleration, not velocity, is paramount. The principle of relativity played only a minor role in the development of classical mechanics; Einstein elevated it to a keystone of dynamics. He extended the principle to include not only the laws of mechanics but also the laws of electromagnetic interaction and, by supposition, all the laws of physics. Furthermore, in his hands the principle of relativity became an important working principle in discovering the correct form of physical laws. We can only surmise the sources of his inspiration, but they must have included the following consideration. If the velocity of light were not a universal constant, that is, if the ether could be detected, then the principle of relativity would fail; a special inertial frame would be singled out, the one at rest in the ether. However, the form of Maxwell's equations, as well as the failure of any experiment to detect motion through the ether, suggests that the speed of light is constant, independent of the motion of the source. Our inability to detect absolute motion, either with light or with newtonian forces, implies that absolute motion has no role in physics.

Whereas most physicists regarded the absence of the ether as a paradox, Einstein saw that its absence preserved the simplicity of the principle of relativity. His view was essentially conservative; he insisted on preserving the principle of relativity which the ether would destroy. Apparently the urge toward simplicity was fundamental to his personality.[1] The special theory of relativity was the simplest way to preserve the unity of classical physics. In fact, as we shall see in the closing chapter, special relativity actually simplifies newtonian thought by combining space and time in a natural fashion from which the various conservation laws follow as a single entity.

The Postulates of Special Relativity

To summarize, the postulates of special relativity are:

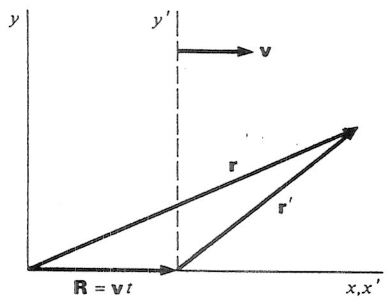
The laws of physics have the same form in all inertial systems.

The velocity of light in empty space is a universal constant, the same for all observers.

The mathematical expression of the special theory of relativity is embodied in the Lorentz transformations—a simple prescription for relating events in different inertial systems. Contrary to the mystique, the mathematics of relativity is quite simple: elementary algebra will suffice. The reasoning is also simple, but it has a deceptive simplicity. We start by looking once more at the Galilean transformations.

[1] Einstein had much in common with Newton. In the second book of his "Principia," Newton states his rules of scientific reasoning. Rule 1 is: "We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances. . . . Nature is pleased with simplicity. . ."

11.4 The Galilean Transformations



Let us review for a moment the newtonian way of viewing an event in different coordinate systems. Consider an inertial system x, y, z, in which we are at rest, and a second inertial system x', y', z', which is translating uniformly in the $+x$ direction with velocity v. For convenience, we take the origins to coincide at $t = 0$, and take the axes to be parallel.

If a particular point in space has coordinates $\mathbf{r} = (x,y,z)$ in our "rest" system, the corresponding coordinates in the moving system are $\mathbf{r}' = (x',y',z')$. These are related by

$$\mathbf{r}' = \mathbf{r} - \mathbf{R},$$

where

$$\mathbf{R} = \mathbf{v}t.$$

Since v is in the x direction, we have

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned} \tag{11.1}$$

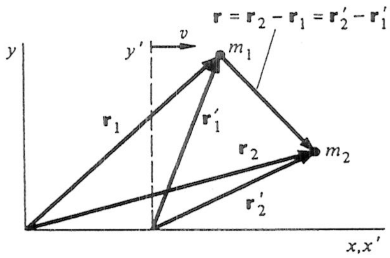
The last equation is listed merely for completeness. It follows from the newtonian idea of an "absolute" time, and it is so taken for granted that it is generally omitted in discussions of classical physics.

Equations (11.1) are known as the *Galilean transformations.* Since the laws of newtonian mechanics hold in all inertial systems, they are unaffected by these transformations. The classical principle of relativity asserts that the laws of mechanics are unchanged by the Galilean transformations. The following example illustrates the meaning of this statement.

Example 11.1 **The Galilean Transformations**

Consider how we might discover the law of force between two isolated bodies from observations of their motion. For example, the problem might be to discover the law of gravitation from data on the elliptical orbit of one of Jupiter's moons. If m_1 and m_2 are the masses of the moon and of Jupiter, respectively, and $\mathbf{r}_1$ and $\mathbf{r}_2$ are their positions relative to an astronomer on the earth, we have

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}(r)$$
$$m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}(r),$$



where we assume that **F**, the force between the bodies, depends only on their separation $r = |\mathbf{r}_1 - \mathbf{r}_2|$. (Including the effect of the sun makes the analysis more cumbersome without changing the conclusions.)

From our data on $\mathbf{r}_1(t)$ we can evaluate $\ddot{\mathbf{r}}_1$, which yields the value of **F**, (or $\mathbf{F}/m_1$, to be more precise). In principle, this is the procedure Newton followed in discovering the law of universal gravitation. Suppose that the data show $\mathbf{F}(r) = -Gm_1m_2\hat{\mathbf{r}}/r^2$.

Now let us consider the problem from the point of view of an astronomer in a spacecraft observatory which is flying by the earth. According to the principle of relativity he must obtain the same force law. The situation is represented in the drawing. x, y is the earthbound system, x', y' is the spacecraft system, and v is the relative velocity.

In the x', y' system the astronomer concludes that the force on m_1 is

$$\mathbf{F}'(r') = m_1\ddot{\mathbf{r}}_1'.$$

However,

$$\mathbf{r}_1 = \mathbf{r}_1' + \mathbf{v}t$$
$$\dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_1' + \mathbf{v}$$
$$\ddot{\mathbf{r}}_1 = \ddot{\mathbf{r}}_1'.$$

Hence,

$$\begin{aligned}\mathbf{F}'(r') &= m_1\ddot{\mathbf{r}}_1' \\ &= m_1\ddot{\mathbf{r}}_1 \\ &= \mathbf{F}(r).\end{aligned}$$

Since $r' = r$, $\mathbf{F}'(r') = \mathbf{F}'(r)$. But we have just shown that $\mathbf{F}'(r') = \mathbf{F}(r)$. Hence,

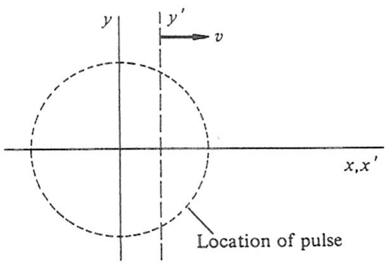
$$\begin{aligned}\mathbf{F}'(r) &= \mathbf{F}(r) \\ &= -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}.\end{aligned}$$

The law of force is identical to the one found on earth. This is what we mean when we say that the two inertial systems are equivalent. If the form of the law, or the value of G, were different in the two systems, we could make a judgment about the speed of a coordinate system by investigating the law of gravitation in that system. The systems would not be equivalent.

Example 11.1 is almost trivial, since the force depends on the separation of the two particles, a quantity which is unchanged (invariant) under the Galilean transformations. In newtonian physics, all forces are due to interactions between particles, interactions which depend on the *relative* coordinates of the particles. Consequently they are invariant under the Galilean transformations.

What happens to the equation for a light signal under the Galilean transformations? The following example shows the difficulty that arises.

Example 11.2 **A Light Pulse as Described by the Galilean Transformations**



At $t = 0$ a pulse of light is emitted isotropically in the x, y system. It travels outward with velocity c. The equation for the wavefront along the x axis is

$$x = ct.$$

In the x', y' system, the equation for the wavefront along the x' axis is

$$\begin{aligned} x' &= x - vt \\ &= (c - v)t, \end{aligned}$$

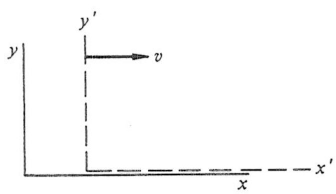
where v is the relative velocity of the two systems.

The x' velocity of the pulse in the x', y' system is

$$\frac{dx'}{dt} = c - v.$$

But this is contrary to the postulate that the speed of light is a universal constant c for all observers. Clearly, the Galilean transformations are inadequate.

11.5 The Lorentz Transformations



Since the Galilean transformations do not satisfy the postulate that the speed of light is a universal constant, Einstein proposed an alternate prescription for describing the same event in different inertial systems. Let us refer once more to our standard systems, the rest system, x, y, z, t and the system x', y', z', t' which moves with velocity v along the positive x axis. The origins coincide at $t = t' = 0$. We take the most general transformation relating the coordinates of a given event in the two systems to be of the form

$$x' = Ax + Bt \qquad 11.2a$$

$$y' = y \qquad 11.2b$$

$$z' = z \qquad 11.2c$$

$$t' = Cx + Dt. \qquad 11.2d$$

The transformations are linear, for otherwise there would not be a simple one-to-one relation between events in the different systems. For instance, a nonlinear transformation would predict acceleration in one system even if the velocity were constant in

TABLE 11.1

EVENT	COORDINATES (x,y,t)	COORDINATES (x',y',t')	TRANSFORMATION LAW	RESULT
Observer in (x,y) sees origin of (x',y') move along x axis with velocity v.	$x = vt$	$x' = 0$	$x' = Ax + Bt$ 11.2a $0 = Avt + Bt$	$B = -Av$
Observer in (x',y') sees origin of (x,y) move along x' axis with velocity $-v$.	$x = 0$	$x' = -vt'$	$x' = A(x - vt)$ 11.2a $t' = Cx + Dt$ 11.2d $A(0 - vt) = -v(0 + Dt)$	$D = A$
A light pulse is sent out from origin along x axis at $t = 0$. Its location is given by:	$x = ct$	$x' = ct'$	$x' = A(x - vt)$ 11.2a $t' = Cx + At$ 11.2d $A(ct - vt) = c(Cct + At)$	$C = -\frac{Av}{c^2}$
A light pulse is emitted along the y axis in (x,y) at $t = 0$. In (x',y') the pulse has components along the x' and y' axes. The velocity of the pulse is c in both systems. Its coordinates are given by:	$x = 0$ $y = ct$	$x'^2 + y'^2 = c^2t'^2$	$x' = A(x - vt)$ 11.2a $y' = y$ 11.2b $t' = A(-vx/c^2 + t)$ 11.2d $A^2(0 - vt)^2 + (ct)^2 = c^2A^2[-(v/c^2)0 + t]^2$	†$A = \frac{1}{\sqrt{1 - v^2/c^2}}$

† In general, $A = \pm 1/\sqrt{1 - v^2/c^2}$. We choose the positive root; otherwise, in the limit $v = 0$ we would find $x' = -x$ rather than $x' = x$ as we require.

the other, clearly an unacceptable property for a transformation between inertial systems. We have assumed that the y' and z' axes are left unchanged by the transformation for reasons of symmetry, which we shall discuss later.

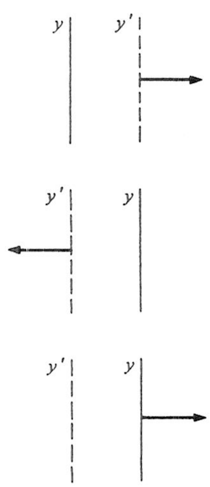
Equations (11.2) contain four unknown constants. To evaluate these we consider four cases in which we know *a priori* how an event appears in the two systems. This is carried out in Table 11.1.

Inserting the results of Table 11.1 into Eq. (11.2) gives

$$\begin{aligned} x' &= \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \frac{1}{\sqrt{1 - v^2/c^2}}\left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{11.3}$$

It is a straightforward matter to solve these equations algebraically for x, y, z, t in terms of x', y', z', t'. Alternatively, we can simply interchange the labels and reverse the sign of v, because the only difference between the systems is the direction of the relative velocity. The result is

$$\begin{aligned} x &= \frac{1}{\sqrt{1 - v^2/c^2}}(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \frac{1}{\sqrt{1 - v^2/c^2}}\left(t' + \frac{vx'}{c^2}\right) \end{aligned} \tag{11.4}$$



(a) (b) (c)

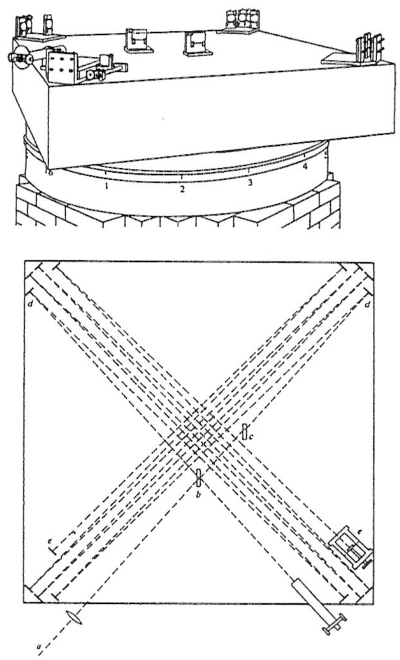
Equations (11.3) and (11.4) are the *Lorentz transformations*, the prescription for relating the coordinates of an event in different inertial systems so as to satisfy the postulates of special relativity. In the following chapters we shall explore their consequences. We conclude the present discussion by explaining the argument for assuming $y = y'$, $z = z'$.

Consider a section of the y and y' axes as shown in figure (a). The y' axis is moving to the right with velocity v.

If we look at the systems from behind the paper, the situation appears as in sketch (b).

Since only relative motion is important, Figure (b) is equivalent to (c). However, (c) is identical to (a) except that y' and y are interchanged. We conclude that the y and y' axes are indistinguishable and $y = y'$. By a similar argument $z = z'$.

Problems



(From *American Journal of Science*, November, 1887.)

11.1 The Michelson-Morley experiment was carried out at the Case School of Applied Science (now Case-Western Reserve University) in 1887. The apparatus was a refined version of the interferometer used by Michelson in his initial search in Berlin during 1881. The interferometer was mounted on a granite slab 5 ft square and 14 in thick resting on a float riding in a mercury-filled trough. The effective length of the interferometer arms was lengthened to 11 m by the use of mirrors. The light source was the yellow line of sodium, $\lambda = 590 \times 10^{-9}$ m. Michelson and Morley found no systematic shift of fringe with direction, although they could have detected a shift as small as one-hundredth fringe.

How does the upper limit to the earth's velocity through the ether set by this experiment compare with the earth's orbital velocity around the sun, 30 km/s? See drawing at left.

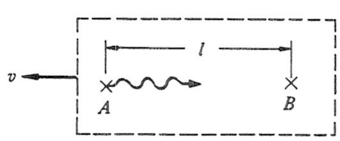
11.2 If the two arms of a Michelson interferometer have lengths l_1 and l_2, show that the fringe shift when the interferometer is rotated by 90° with respect to the velocity v through the ether is

$$N = \frac{l_1 + l_2}{\lambda} \frac{v^2}{c^2},$$

where λ is the wavelength of the light.

11.3 The Irish physicist G. F. FitzGerald and the Dutch physicist H. A. Lorentz independently tried to explain the null result of the Michelson-Morley experiment by the following hypothesis: motion of a body through the ether sets up a strain which causes the body to contract along the line of motion by the factor $1 - \frac{1}{2}v^2/c^2$. Show that this hypothesis accounts for the absence of a fringe shift in the Michelson-Morley experiment. (The hypothesis was disproved in 1932 by R. J. Kennedy and E. M. Thorndike, who repeated the Michelson-Morley experiment with an interferometer having arms of different lengths.)

11.4 The Michelson-Morley experiment is known as a second order experiment because the observed effect depends on $(v/c)^2$. Consider the following first order experiment.



At time $t = 0$, observer A sends a signal to observer B a distance l away. B records the arrival time. Assume that the system is moving through the ether with speed v in the direction shown. Suppose that the laboratory is then rotated 180° with respect to the velocity, reversing the positions of A and B. At time $t = T$, A sends a second signal to B.