**Physics 22** **Final Exam** **Spring 2010**

SOLUTIONS

**Instructions:**

First, this is a closed book exam. That means no notes, no books, and no use of a calculator.

Second, do not open your exam until I say it is OK to begin.

Third, while waiting to begin, please PRINT your name CLEARLY on your blue book.

**Advice:**

Please start each problem on a new page in your blue book, and please DO NOT do algebra in your head. First, you WILL make a mistake, and second you WILL NOT get any partial credit. Write it down, line-by-line as you do it. Also, please try to be NEAT, REALLY NEAT. This will help your grade, although in a perfect world maybe it shouldn't.

There are 8 straightforward problems (45 points total) and 4 regular ones (55 points total). I think most of you can do rather well with the regular problems, if you give yourself time to work on them. My strategy would be to work on them first for about an hour, and then tackle the straightforward ones for another hour. I would use the last hour wherever you think best, based on how things have been going.

If you finish early, check your work instead of leaving. You can always find a mistake if you really try. I think the average will be pretty reasonable, so you don't want to be losing points for careless mistakes.

**Straightforward Problems: These problems will be graded on two considerations. First, have you written down the relevant equation(s) correctly? Second, have you plugged the numbers in correctly? Put in the numbers and skip the arithmetic.**

1. Two harmonic waves, given by $y_1(x,t) = y_O Sin(2\pi x/5 - 30\pi t)$ and $y_2(x,t) = y_O Sin(2\pi x/6 - 25\pi t)$ are traveling on a string. The units are mks.

a. (2 points) With what speed are the wave fronts for each of the waves moving?

$v_p = \omega/k$, $v_{p1} = 30\pi\, s^{-1}/(2\pi/5)m^{-1} = 75\, m/s$

$v_{p2} = 25\pi s^{-1}/(2\pi/6\, m^{-1}) = 75 m/s$

b. (2 points) With what velocity (not speed) is the string located at $x = 5$ m moving at $t =$ 0 s?

$v = \frac{\partial y}{\partial t} = -30\pi y_o \cos(2\pi x/5 - 30\pi t) - 25\pi y_o \cos(\frac{2\pi x}{6} - 25\pi t)$

$x = 5m,\ t=0 \Rightarrow v = -30\pi y_0 \cos[2\pi] - 25\pi y_o \cos[2\pi \times 5/6]$

c. (2 points) With what frequency (in Hz, not radians/sec) do the two wave beat?

$\nu_B = |\nu_1 - \nu_2| = \frac{|\omega_1 - \omega_2|}{2\pi} = \frac{30\pi - 25\pi}{2\pi} = \frac{5}{2} = 2.5\ Hz$

2. A source emits spherical harmonic sound waves in 3 dimensions in a medium having a bulk modulus of $10^5$ Pascal. The waves result in a pressure perturbation of 0.01 Pascal at a distance 10 m from the source.

a. (2 points) What pressure perturbation do the waves produce 2 m from the source?

$I \propto \frac{1}{r^2} \propto \delta P^2 \Rightarrow \delta P \propto \frac{1}{r}$, so 5x Larger $\delta P_{2m} = 0.01 \times 5 = 0.05\ Pa$

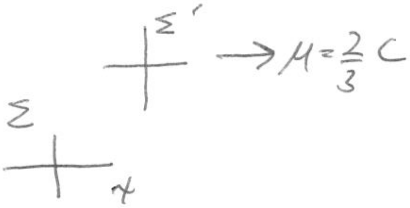
b. (2 points) What fractional density perturbation do the waves produce 10 m from the source?

$\frac{\delta \rho}{\rho} = \frac{\delta P}{B} = \frac{0.01\ Pa}{10^5 Pa} = 10^{-7}\ Pa$

c. (2 points) If the medium has a density of 10 kg/m³, with what speed do the waves travel?

$v_p = \sqrt{B/\rho} = \sqrt{\frac{10^5 Pa}{10\ kg/m^3}} = \sqrt{10^4}\ m/s = 100\ m/s$

3. (5 points) If an event occurs at position $x = 3$ m at time $t = 10^{-7}$ s in one frame of reference, at what position and time will this event occur in a frame moving toward positive x at a speed of $2c/3$, given that the origins of the two reference frames coincided at $t = 0$?




$$x' = \gamma(x - ut) = \frac{1}{\sqrt{1 - 4/9}}\left(3 - \frac{2}{3}c \times 10^{-7}\right) m$$

$$x' = \frac{3}{\sqrt{5}}\left[3 - 20\right] = -\frac{3 \cdot 17}{\sqrt{5}} \ m$$

$$t' = \gamma\left[t - \frac{ux}{c^2}\right] = \frac{3}{\sqrt{5}}\left[10^{-7} s - \frac{2}{3} \times \frac{3 \ m}{3 \times 10^8 \ m/s}\right]$$

$$t' = \frac{3}{\sqrt{5}}\left[10^{-7} - \frac{2}{3} \times 10^{-8}\right] s = \frac{3 \times 10^{-7}}{\sqrt{5}}\left[1 - 0.2/3\right] s$$

4. (5 points) Given that a certain metal has a thermal conductivity of 200 W/m K, how many kg of ice will melt in an hour if a square rod made of that alloy, which is 2 cm by 2 cm in cross section, and 50 cm long, has one end held at the temperature of boiling water and the other end is in an ice/water mixture? [Hint: Water has a latent heat of fusion of 333,000 J/kg]

$$H = \dot{Q} = \frac{kA}{L}(T_2 - T_1) = 200 \frac{W}{mK} \times \frac{(0.02m)^2}{0.5m} \times 100 \ K$$

$$H = \frac{2 \times 10^4 \times 4 \times 10^{-4}}{0.5} \ W = 16 \ \text{Watts}$$

In one hour $Q = 16 \frac{J}{s} \times 3600 s$ enters the ice bath

this will melt a mass $m$ of ice, such that

$$333{,}000 \frac{J}{kg} \times m \ (kg) = 16 \times 3600 \ J$$

so $m = \frac{16 \times 3.6}{333} \ kg$

5. (5 points) Steel has a thermal expansion coefficient of $1.1\times10^{-5}\ \mathrm{K}^{-1}$, while that of brass is $1.9\times10^{-5}\ \mathrm{K}^{-1}$. What length of brass rod would be needed to expand by the same amount for any given temperature change as a 25 cm long steel rod?

$$\delta L_B = L_B \alpha_B \Delta T, \quad \delta L_S = L_S \alpha_S \Delta T$$

$$\delta L_B = \delta L_S \Rightarrow L_B = L_S \frac{\alpha_S}{\alpha_B} = 25\text{ cm} \times \frac{1.1}{1.9}$$

6. (5 points) Given the Maxwell speed distribution for molecules of mass $m$ in a uniform gravitational field

$$N(v,z) = 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_BT}}\left(\frac{mg}{k_BT}\right)e^{-\frac{mgz}{k_BT}}$$

, write down the double integral that equals the fraction (not the number) of molecules that will be found to have altitudes between $\frac{k_BT}{mg}$ and $\frac{3k_BT}{mg}$, and which will also have speeds between 100 and 300 m/s. Don't bother to do any integrals.

$$N\left(100 < v < 300, \frac{k_BT}{mg} < z < \frac{3k_BT}{mg}\right) = \int_{100}^{300}\int_{k_BT/mg}^{3k_BT/mg} N(v,z)\, dz\, dv$$

$$f = 4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2}\left(\frac{mg}{k_BT}\right)\int_{100\text{ m/s}}^{300\text{ m/s}} v^2 e^{-mv^2/2k_BT}\, dv \int_{k_BT/mg}^{3k_BT/mg} e^{-mgz/k_BT}\, dz$$

if we like, we can do the $\int$ on $z$.

$$f = 4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2}\left[e^{-1}-e^{-3}\right]\int_{100\text{ m/s}}^{300\text{ m/s}} v^2 e^{-mv^2/2k_BT}\, dv$$

7. Three moles of an ideal diatomic gas with specific heat ratio 1.4 expands reversibly and isothermally at T=450 K from an initial volume to a final volume that is 4 times the initial volume.

a. (2 points) How much work is done by the gas during the expansion?

$$W_{\substack{BY\\GAS}} = \int_{V_i}^{V_f} P\,dV = \int \frac{nRT}{V}\,dV = nRT\int_{V_i}^{V_f}\frac{dV}{V} = nRT\ln 4$$

$$= 3 \times 8.31\ \mathrm{J/K} \times 450\,\mathrm{K}\ \ln 4$$

b. (2 points) How much heat flows into the gas during the expansion?

$\Delta E_{INT} = 0$ (I.G. $E_{INT} = E(T \text{ only})$) $\therefore Q_{IN} = W_{OUT}$

$\therefore Q_{IN} = 3 \times 8.31 \times 450 \ln 4$ Joules

c. (2 points) By how much does the internal energy of the gas change during the expansion?

$E_{int} = E(T)$ ideal gas $\therefore$ $\Delta T = 0 \Rightarrow \Delta E_{INT} = 0$

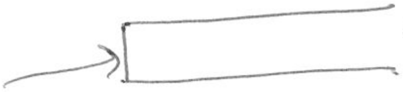
d. (2 points) By how much does the entropy of the gas change during the expansion?

$\Delta S = \int \frac{đQ}{T} = \frac{1}{T} Q$ for isothermal

reversible

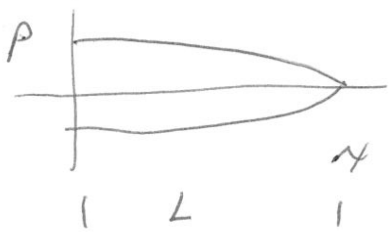
$\therefore \Delta S = +3 \times 8.31 \times \ln 4$ J/K

8. (5 points) A 10 m long organ pipe is closed at one end and open at another. What is its fundamental resonant frequency? Recall that the speed of sound in air is 343 m/s.



P node at open end

Displacement node at closed end ⇒ P anti-node




nodes are $\frac{\lambda}{2}$ apart, so

$\frac{\lambda}{4} = L$, $\lambda = 4L = \frac{v}{f_0}$

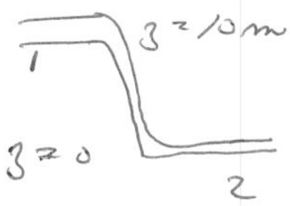
$\therefore f_0 = \frac{v}{4L} = \frac{343\ \mathrm{m/s}}{40\ \mathrm{m}}$

$f_0 = \frac{343}{40}$ Hz

## Normal Problems

9. (10 points) A horizontal pipe contains water flowing at 4 m/s at a pressure of 2 atmospheres. The pipe gradually narrows to half its original diameter while also going downhill a vertical distance of 10 m, before becoming horizontal again. Find the pressure in the lower horizontal section, assuming the flow is laminar and viscosity can be neglected. Recall that one atmosphere is $1.0 \times 10^5$ Pascal, and the density of water is 1,000 kg/m$^3$.

Bernoulli's Principle Applies




So $P_1 + \frac{1}{2} \rho v_1^2 + \rho g z_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g z_2$

Also what goes in comes out!

i.e. $A_1 v_1 = A_2 v_2$ $\therefore$ $\pi \cancel{r_1^2} v_1 = \pi \left(\frac{\cancel{r_1}}{2}\right)^2 v_2$

$v_2 = 4 v_1$, $z_2 = 0$, $z_1 = 10\text{ m}$, $P_1 = 2\text{ atm}$

So $2 \times 10^5\text{ Pa} + \frac{1}{2} \times 1000 [v_1^2] + \rho g \times 10\text{ m} = P_2 + \frac{1}{2} \rho 16 v_1^2$

$$\begin{array}{r} 298 \\ -120 \\ \hline 178 \end{array}$$

$P_2 = 2 \times 10^5\text{ Pa} - \frac{1}{2} \times 1000 \times 15 \times 16 + 1000 \times 9.8 \times 10\text{ Pa}$

$= 2 \times 10^5\text{ Pa} - 120,000\text{ Pa} + 98,000\text{ Pa} = 178,000\text{ Pa} = 1.78\text{ Atm}$

10. (10 points) Two submarines are moving directly toward each other. Each submarine is traveling at 10 m/s, while the speed of sound in water is 1500 m/s. One submarine emits a sonar signal (a sound wave) at 2.5 kHz, which is reflected from the front of the second submarine. The reflected sound waves are subsequently detected by the first submarine. Find the frequency of the reflected waves as measured by the first submarine. [Hint: There is more than one source in this situation.]

The two moving sources are the first sub which launched the sonar wave, and the second sub which acts like a moving source, when it reflects the waves. For a source and "observer" moving toward each other we have

$f' = f_0 \frac{[1 + u_0/v]}{1 - u_s/v}$, where $u_0 = u_s = 10$ m/s here and $v = 1500$ m/s

The second sub acts as a source at $f'$, with the first one now being the observer moving toward it.

$\therefore f'' = f_0 \frac{[1 + 10/1500]^2}{[1 - 10/1500]^2} = 2.5\text{ kHz} \times \left[\frac{1 + 1/150}{1 - 1/150}\right]^2$

11. (10 points) Use what you have learned about ideal gasses to derive the equation giving the speed of sound in an ideal *monatomic* gas composed of atoms of mass *m*, which is in equilibrium at temperature *T* (in Kelvin). Recall that it is the adiabatic bulk modulus $-V\left.\frac{\partial P}{\partial V}\right)_S$ that determines the sound speed.

$PV^\gamma = \text{const} \Rightarrow dP\,V^\gamma + \gamma P V^{\gamma-1} dV = 0$

so $-\left.\frac{\partial P}{\partial V}\right)_S = \frac{\gamma P}{V} \Rightarrow B = \gamma P$. So $v = \sqrt{\frac{\gamma P}{\rho}}$

monatomic $\Rightarrow \gamma = \frac{5/2}{3/2} = \frac{5}{3}$

and $P = \frac{N}{V} k_B T = \rho \frac{k_B T}{m} \Rightarrow v = \sqrt{\frac{5}{3}\frac{k_B T}{m}}$.

12. Consider a quantum system that has only two states, one with energy $-E_O$, and one with energy $+E_O$. Consider a thermodynamic system consisting of a large number $N$ of such quantum systems in thermal equilibrium at temperature T (in Kelvin).
[Hint: This is an easy problem.]

a. (5 points) Find the fraction of the quantum systems that will be in the first excited state, i.e. have energy $+E_O$. (Hint: This fraction is identical to the probability that any particular quantum system will be in that state.)

$P(+E_0 \text{ state}) = C e^{-E_0/k_BT}$ $\quad C = ?$ $\quad P(-E_0) + P(E_0) = 1$

so $C e^{E_0/k_BT} + C e^{-E_0/k_BT} = 1 \Rightarrow C = \frac{1}{e^{E_0/k_BT} + e^{-E_0/k_BT}}$

$\therefore P(+E_0 \text{ state}) = \frac{e^{-E_0/k_BT}}{e^{E_0/k_BT} + e^{-E_0/k_BT}}$

Note! $T \to \infty \Rightarrow P \to \frac{1}{1+1} = \frac{1}{2}$ ✓

b. (10 points) Find the average energy of the thermodynamic system consisting of the N quantum systems.

$\langle E \rangle = +E_0 P(+E_0) + (-E_0) P(-E_0)$

$= \frac{E_0 e^{-E_0/k_BT}}{e^{E_0/k_BT} + e^{-E_0/k_BT}} - \frac{E_0 e^{E_0/k_BT}}{e^{E_0/k_BT} + e^{-E_0/k_BT}}$

IN CASE YOU ARE FAMILIAR WITH HYPERBOLIC FCNS

$= -E_0 \frac{[e^{E_0/k_BT} - e^{-E_0/k_BT}]}{[e^{E_0/k_BT} + e^{-E_0/k_BT}]} = -E_0 \cdot \frac{2\sinh(E_0/k_BT)}{2\cosh(E_0/k_BT)}$

$= -E_0 \tanh(E_0/k_BT)$

13. Two sources produce spherical harmonic waves in 3-dimensions, and they are positioned on the *y*-axis a distance *d* above and below the *x*-axis, as shown. Each source, when driven alone, produces a wave of amplitude $A_o$ at the origin, a distance *d* from itself. The wave speed is given as $V_o$. The two sources are driven at the same frequency *f*, but exactly ***out of phase*** with each other.

a. (5 points) Find the lowest driving frequency for which the disturbance at the point $(x = 2d, y = d)$ directly across from the upper source will be a minimum.

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b. (5 points) What will be the ratio of the intensity at the point $(x = 2d, y = d)$ for this lowest frequency minimum to the intensity of one source alone at a distance of *d*?

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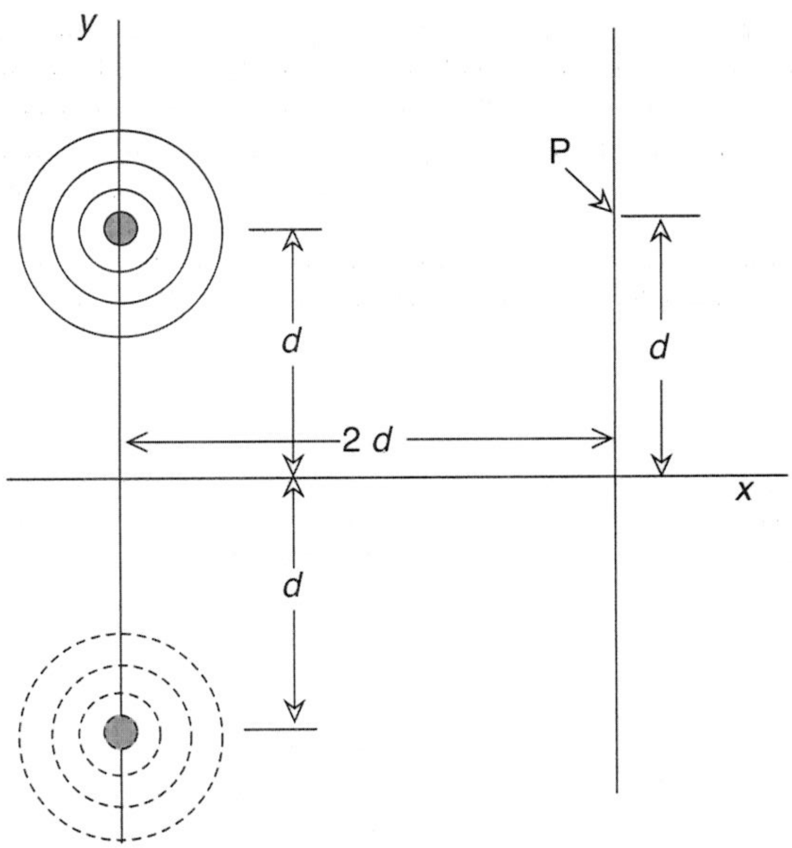
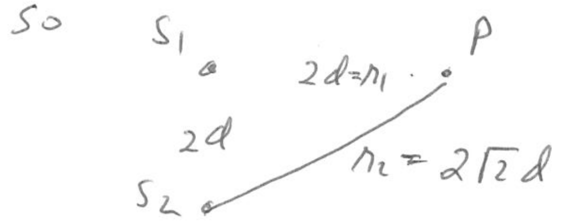


13 a. For minimum amplitude, the difference in the distances to the two sources must be a full wavelength, or two or three, etc, because the sources are already out of phase, i.e. shifted by $\frac{\lambda}{2}$.

So




$$r_1 = 2d$$
$$r_2 = 2\sqrt{2}\,d$$

$$|r_2 - r_1| = 2d[\sqrt{2} - 1] = \lambda_1$$

$$\lambda_1 = \frac{V_0}{f_1} \Rightarrow \boxed{f_1 = \frac{V_0}{2d[\sqrt{2}-1]}}$$

is the lowest frequency for which a minimum will occur.

13 b. The minimum amplitude will <u>not</u> be zero, because $S_2$ is further away. For $S_1$, the amplitude will be decreased by a factor of 2 from its value at $r = d$, ie $\frac{A_0}{2}$. This is because $I = \frac{P}{A} \propto \frac{1}{r^2} \propto A^2$

So $A \propto \frac{1}{r}$. For $A_2$ we have $A_2 = \frac{A_0}{2\sqrt{2}}$, which is weaker, so it won't be able to fully cancel $S_1$.

$$\text{So } A_{TOTAL} = A_1 + A_2 = \frac{A_0}{2}\left[1 - \frac{1}{\sqrt{2}}\right]$$

Thus the intensity will be $\propto \frac{A_0^2}{4}\left[1 - \frac{1}{\sqrt{2}}\right]^2$ vs $A_0^2$ at $r = d$ for $S_1$ alone.

ie $\frac{I_{MIN}}{I_1(r=d)} = \frac{[1 - 1/\sqrt{2}]^2}{4} \simeq \frac{(1-0.7)^2}{4} \simeq \frac{.09}{4}$, weak, but NOT 0.