
Phys 250 Quantum Optics, Homework #1, Solutions

For 1-3, just follow the text.

4) Give an explicit expression for the conserved quantities corresponding to the invariance of the Lagrangian under Lorentz boosts.

Equations (3.41) - (3.46) in the reading give the general outline, but it's not as straightforward as the other problems. Start with the Lorentz boost:

$$\begin{aligned}\delta x^\mu &= \epsilon^\mu_\nu x^\nu \\ \epsilon^{\mu\nu} &= -\epsilon^{\nu\mu}\end{aligned}$$

Making use of the anti-symmetry and (3.42):

$$\begin{aligned}\delta x^\mu &= \frac{1}{2} \epsilon^{\rho\sigma} \left[\underbrace{(X^\mu_{\rho\sigma} + X^\mu_{\sigma\rho})}_0 - (X^\mu_{\rho\sigma} - X^\mu_{\sigma\rho}) \right] \\ &= -\frac{1}{2} \epsilon^{\rho\sigma} (X^\mu_{\rho\sigma} - X^\mu_{\sigma\rho}) \\ &= -\frac{1}{2} \epsilon_{\rho\sigma} (\delta^\mu_\rho x^\sigma - \delta^\mu_\sigma x^\rho) \\ &= -\frac{1}{2} (\epsilon_{\mu\sigma} x^\sigma - \epsilon_{\rho\mu} x^\rho)\end{aligned}$$

The associated Noetherian conserved current is:

$$\begin{aligned}J^{\mu\rho\sigma} &= -\Theta^\mu_\kappa X^{\kappa\rho\sigma} \\ &= -\frac{1}{2} (\Theta^{\mu\rho} x^\sigma - \Theta^{\mu\sigma} x^\rho)\end{aligned}$$

Let us then define:

$$M^{\mu\nu} = \int 2J^{0\mu\nu} d^3x$$

Making use of the fact that $\Theta^{0i} = p^i$:

$$\begin{aligned}M^{ij} &= -\int (\Theta^{0i} x^j - \Theta^{0j} x^i) d^3x \\ &= -\int (p^i x^j - p^j x^i) d^3x \\ &= -\int (\vec{p} \times \vec{x})_k \varepsilon^{ijk} d^3x\end{aligned}$$

In other words the off diagonal elements are the angular momentum of the field.

Additionally, noting that $\Theta^{00} = E$:

$$\begin{aligned}M^{0j} &= -\int (\Theta^{00} x^j - \Theta^{0j} x^0) d^3x \\ &= -\int (Ex^j - p^j t) d^3x\end{aligned}$$

Which is the field's center of mass minus its motion (momentum times time).