6.1 is just a special case of 6.2!

6.2 $U = \exp\left[i\frac{\theta}{2} (a_0^+ a_1 + a_1^+ a_0)\right]$

Note: $e^{i\lambda\hat{A}}\hat{B}e^{-i\lambda\hat{A}} = \hat{B} + i\lambda[\hat{A},\hat{B}] + \frac{(i\lambda)^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$

So... $\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = U^+ \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} U$

$$= \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} - i\frac{\theta}{2}\left[\hat{A}, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\right] - \frac{\theta^2}{8}\left[\hat{A},\left[\hat{A},\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\right]\right] + \dots$$

where $\hat{A} = a_0^+ a_1 + a_1^+ a_0$

$$\left[a_0^+ a_1 + a_1^+ a_0, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\right] = a_0^+\cancel{\left[a_1, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\right]}^{0} + \left[a_0^+, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\right]a_1 +$$

$$a_1^+\cancel{\left[a_0, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\right]}_{0} + \left[a_1^+, \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\right]a_0$$

$$= -\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

$$\Rightarrow \left[\hat{A},\left[\hat{A},\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\right]\right] = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

$$\Rightarrow U\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}U^+ = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\left[1 - \frac{(\theta/2)^2}{2} + \dots\right] + i\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}\left[\frac{\theta}{2} - \frac{(\frac{\theta}{2})^3}{3!} + \dots\right]$$

$$\boxed{\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\cos(\theta/2) + i\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}\sin(\theta/2)}$$

for $\theta = \pi/2$, $\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a_0 + i a_1 \\ a_1 + i a_0 \end{pmatrix}$ (problem 6.1)

from (6.8) $\begin{pmatrix} t' \\ r' \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ i\sin(\theta/2) \end{pmatrix}$

$$\begin{pmatrix} t \\ r \end{pmatrix} = \begin{pmatrix} t' \\ r' \end{pmatrix}$$

$$\Rightarrow \boxed{\theta = 2\arctan\left(\frac{|r'|}{|t'|}\right)}$$

6.6 $|\alpha\rangle_0 |\beta\rangle_1 = D_0(\alpha) D_1(\beta) |0\rangle_0 |0\rangle_1$

$$D_n = \exp[\alpha a_n^+ - \alpha^* a_n]$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_0^+ \\ a_1^+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_2 - i a_3 \\ -i a_2 + a_3 \\ a_2^+ + i a_3^+ \\ i a_2^+ + a_3 \end{pmatrix}$$

$$D_0(\alpha) = \exp(\alpha a_0^+ - \alpha^* a_0)$$

$$= \exp\left[\frac{\alpha}{\sqrt{2}}(a_2^+ + i a_3^+) - \frac{\alpha^*}{\sqrt{2}}(a_2 - i a_3)\right]$$

$$= \exp\left[\frac{\alpha}{\sqrt{2}}(a_2^+) - \frac{\alpha^*}{\sqrt{2}}(a_2)\right] \exp\left[\frac{i\alpha}{\sqrt{2}}(a_3^+) - \frac{(i\alpha)^*}{\sqrt{2}}(a_3)\right]$$

$$= D_2(\alpha/\sqrt{2})\ D_3(i\alpha/\sqrt{2})$$

similarly: $D_1(\beta) = D_2(i\alpha/\sqrt{2})\, D_3(\alpha/\sqrt{2})$

$$\Rightarrow |\alpha\rangle_0 |\beta\rangle_1 \xrightarrow{BS} D_2(\alpha/\sqrt{2})\, D_2(i\beta/\sqrt{2})\, D_3(i\alpha/\sqrt{2})\, D_3(\beta/\sqrt{2}) \times |0\rangle_2 |0_3\rangle$$

$$\boxed{|\alpha\rangle_0 |\beta\rangle_1 \xrightarrow{BS} \left|\tfrac{1}{\sqrt{2}}(\alpha + i\beta)\right\rangle_2 \left|\tfrac{1}{\sqrt{2}}(i\alpha + \beta)\right\rangle_3}$$

(ignoring an overall phase factor which is not significant!)

This state is not entangled.

6.8 $|N\rangle_0|N\rangle_1 = \frac{1}{N!}(a_0^\dagger a_1^\dagger)^N|0\rangle_0|0\rangle_1$

$$\xrightarrow{BS} \frac{1}{N!}\left[\frac{1}{2}(a_2^\dagger + i a_3^\dagger)(i a_2^\dagger + a_3)\right]^N |0\rangle_2|0_3\rangle$$

$$\to \frac{1}{2^N N!}\left[i a_2^{\dagger 2} - \cancel{a_3^\dagger a_2^\dagger} + \cancel{a_3^\dagger a_2^\dagger} + i a_3^{\dagger 2}\right]^N |0\rangle_2|0\rangle_3$$

$$\to \frac{i^N}{2^N N!}\left[a_2^{\dagger 2} + a_3^{\dagger 2}\right]^N |0\rangle_2|0\rangle_3$$

$$\to \left(\frac{i}{2}\right)^N \frac{1}{N!}\sum_{k=0}^{N}\binom{N}{k}(a_2^{\dagger 2})^{N-k}(a_3^{\dagger 2})^k |0\rangle_2|0\rangle_3$$

$$\to \sum_k \left(\frac{i}{2}\right)^N \frac{N!}{N!\,k!\,(N-k)!}\sqrt{(2N-2k)!\,(2k)!}\;|2N-2k\rangle_2|2k\rangle_3$$

$$\to \sum_k i^N\left[\left(\frac{1}{2}\right)^{2N}\frac{(2k)!}{(k)!\,(2k-k)!}\,\frac{(2N-2k)!}{(N-k)!\,(N-k)!}\right]^{1/2}|2N-2k\rangle_2|2k\rangle_3$$

$$\to \sum_k i^N\left[\binom{2k}{k}\binom{2N-2k}{N-k}\left(\frac{1}{2}\right)^{2N}\right]^{1/2}|2k\rangle_2|2N-2k\rangle_3$$

The i^N term is an overall phase, and is not significant.

6.11 from 6.2:
$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \cos\phi + i \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \sin\phi$$
$$(\phi = \theta/2)$$

$$\Rightarrow \begin{pmatrix} a_0^+ \\ a_1^+ \end{pmatrix} = \begin{pmatrix} a_2^+ \\ a_3^+ \end{pmatrix} \cos\phi + i \begin{pmatrix} a_3^+ \\ a_2^+ \end{pmatrix} \sin\phi .$$

$$|0\rangle|1\rangle \xrightarrow{BS1} |0\rangle|1\rangle\cos\phi_1 + i|1\rangle|0\rangle\sin\phi_1$$

$$\Rightarrow |0\rangle|1\rangle \xrightarrow{BS1}\xrightarrow{BS2} \cos\phi_1\left(|0\rangle|1\rangle\cos\phi_2 + i|1\rangle|0\rangle\sin\phi_2\right) +$$
$$\sin\phi_1\left(i|1\rangle|0\rangle\cos\phi_2 - |0\rangle|1\rangle\sin\phi_2\right)$$
$$\rightarrow\rightarrow (\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2)|0\rangle_{D_1}|1\rangle_{D_2} +$$
$$i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)|1\rangle_{D_1}|0\rangle_{D_2}$$

We want all of the light to go to one detector (D_1) if no object is present.

Thus: $\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 = 0 \quad (= \sqrt{P(D_2)})$

$$\Rightarrow \phi_1 = \pi/2 - \phi_2 \quad \text{or} \quad \theta_1 = \pi - \theta_2$$

(Note that this is satisfied for 50:50 BS.)

What if an object is present?

$$|0\rangle|1\rangle \xrightarrow{BS1} |0\rangle|1\rangle\cos\phi_1 + \cancel{i|1\rangle|0\rangle\sin\phi_1}$$
$$|0\rangle|0\rangle\sin\phi_1$$

$$\xrightarrow{BS1}\xrightarrow{BS2} \cos\phi_1\left(|0\rangle|1\rangle\cos\phi_2 + i|1\rangle|0\rangle\sin\phi_2\right) + |0\rangle|0\rangle\sin\phi_1$$

$$\Rightarrow P(D_2) = [\cos\phi_1\cos\phi_2]^2 = (\cos\phi_1\sin\phi_1)^2$$

which is the chance of detecting the object.

It has a maximum of 1/4 at $\phi = \frac{\pi}{4}(2n+1)$ or
$$\theta = \frac{\pi}{2}(2n+1)$$

Thus a 50:50 BS is ideal.

(Note: I ignored the phase shifter in one arm, but it is easy to show that the ideal phase is 0, as in the 50:50 case.)