

1. 4.5 We start with (4.155):

$$|\psi(t)\rangle = \sum_n C_n \left[\cos(\phi_n/2) |n, +\rangle e^{-i\Omega_n t/2} - \sin(\phi_n/2) |n, -\rangle e^{+i\Omega_n t/2} \right] e^{+i\omega t}$$

(Note: $E_{\pm}(n) = (n + \frac{1}{2})\hbar\omega \pm \hbar\Omega_n(\Delta)/2$;

the book has a typo in 4.145!)

$$|n, +\rangle = \cos(\phi_n/2) |e\rangle |n\rangle + \sin(\phi_n/2) |g\rangle |n+1\rangle$$

$$|n, -\rangle = -\sin(\phi_n/2) |e\rangle |n\rangle + \cos(\phi_n/2) |g\rangle |n+1\rangle$$

$$\text{w/ } \phi_n = \tan^{-1}\left(\frac{\Omega_n(\omega)}{\Delta}\right)$$

$$\Rightarrow |\psi(t)\rangle = \sum_n C_n \left[(\cos^2(\phi_n/2) e^{-i\Omega_n t/2} + \sin^2(\phi_n/2) e^{+i\Omega_n t/2}) |e\rangle |n\rangle + \underbrace{\sin(\phi_n/2) \cos(\phi_n/2) (e^{-i\Omega_n t/2} - e^{+i\Omega_n t/2})}_{-2i \sin(\Omega_n t/2)} |g\rangle |n+1\rangle \right] e^{+i\omega t}$$

$$\omega(t) = P_e(t) - P_g(t) = 1 - 2P_g(t)$$

$$P_g(t) = \sum_n | \langle n+1 | g | \rangle | \psi(t) \rangle |^2 = \sum_n | \langle n+1 | g | \rangle | \psi(t) \rangle |^2$$

$$= \sum_n |C_n|^2 \underbrace{\sin^2(\phi_n/2) \cos^2(\phi_n/2)}_{\frac{4\lambda^2(n+1)}{\Omega_n^2}} 4 \sin^2(\Omega_n t/2)$$

$$\text{from (4.149): } \frac{1}{4} \frac{\Omega_n^2 - \Delta^2}{\Omega_n^2} = \frac{\lambda^2(n+1)}{\Omega_n^2}$$

$$P_g(t) = \sum_n |C_n|^2 \frac{4\lambda^2(n+1)}{\Omega_n^2} \sin^2(\Omega_n t/2)$$

$$\Rightarrow \omega(t) = \sum_n |C_n|^2 \left[1 - \frac{8\lambda^2(n+1)}{\Omega_n^2} \sin^2(\Omega_n t/2) \right]$$

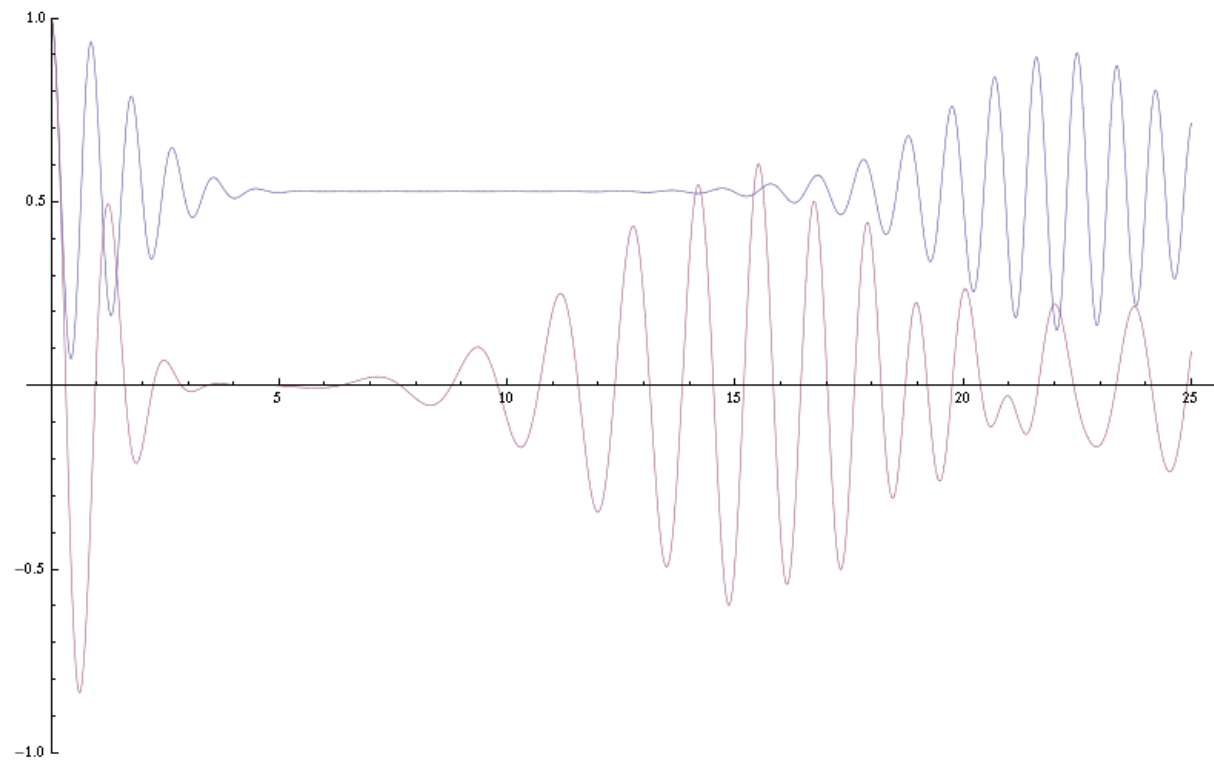
This agrees w/ 4.123 for $\Delta \gg 0$!

$$\Rightarrow \boxed{\omega(t) = \sum_n \left(1 - \frac{4\lambda^2(n+1)}{\Omega_n^2} \right) + |C_n|^2 \frac{4\lambda^2(n+1)}{\Omega_n^2} \cos(\Omega_n t)}$$

(graph on next page)

$$W = \text{Exp}[-nb] \frac{nb^n}{n!} \left(1 - \frac{8\lambda^2 (n+1)}{\Delta^2} \sin[\Delta t / 2]^2 \right) /. \Delta \rightarrow \sqrt{\Delta^2 + 4\lambda^2 (n+1)} ;$$

Plot[{Sum[W /. {λ → 1, Δ → 5, nb → 5}, {n, 0, 20}], Sum[W /. {λ → 1, Δ → 0, nb → 5}, {n, 0, 20}]], {t, 0, 25}, PlotRange → {-1, 1}]
 (*Plot shows Δ = {0, 5}*)



$$\Omega_n = \sqrt{\Delta^2 + 4\lambda^2(n+1)} \cong \Omega_{\bar{n}} + \frac{2\lambda^2(n-\bar{n})}{\Omega_{\bar{n}}} + \mathcal{O}(n-\bar{n})^2$$

$$\omega'(t) = \sum_n |c_n|^2 \frac{2\lambda^2(n+1)}{\Omega_n^2} \left(e^{i\Omega_n t} + e^{-i\Omega_n t} \right)$$

(just the oscillating part of $\omega(t)$)

$$\cong e^{-\bar{n}} \sum_n \frac{\bar{n}^n}{n!} \underbrace{\frac{2\lambda^2(n+1)}{\Omega_n^2}}_{\cong \frac{2\lambda^2(\bar{n}+1)}{\Omega_{\bar{n}}^2} \text{ for } \bar{n} \gg 1} \left(e^{i(\Omega_{\bar{n}} + \frac{2\lambda^2\bar{n}}{\Omega_{\bar{n}}} - \frac{2\lambda^2\bar{n}}{\Omega_{\bar{n}}})t} + e^{-\dots} \right)$$

$$\cong \frac{2\lambda^2(\bar{n}+1)}{\Omega_{\bar{n}}^2} \text{ for } \bar{n} \gg 1$$

$$\cong \frac{2e^{-\bar{n}} \bar{n}^{\bar{n}}}{\Omega_{\bar{n}}^2} \exp \left[i t \left(\Omega_{\bar{n}} + \frac{2\lambda^2\bar{n}}{\Omega_{\bar{n}}} \right) + \underbrace{\bar{n} e^{\frac{-2i\lambda^2}{\Omega_{\bar{n}}} t}}_{\cong \bar{n} - \frac{2i\lambda^2}{\Omega_{\bar{n}}} \bar{n} t - \frac{2\lambda^4}{\Omega_{\bar{n}}^2} \bar{n} t^2 + \dots} \right] + \exp[-\dots]$$

by comparing with 4.133 - 4.135, we see the quadratic term gives the decay time:

$$\boxed{t_c = \frac{\Omega_{\bar{n}}}{\sqrt{2\bar{n}} \lambda^2}}$$

$$\text{for } \Delta=0, \Omega_{\bar{n}} = 2\lambda\sqrt{\bar{n}+1} \Rightarrow t_c = \frac{\sqrt{2}}{\lambda} \sqrt{\frac{\bar{n}+1}{\bar{n}}}$$

which agrees w/ (4.136)

$$\text{from (4.137): } \underbrace{(\Omega(\bar{n}+1) - \Omega(\bar{n}))}_{\cong \frac{2\lambda^2}{\Omega_{\bar{n}}}} t_R = 2\pi k$$

$$\cong \frac{2\lambda^2}{\Omega_{\bar{n}}} \text{ from expansion above}$$

$$\Rightarrow \boxed{t_R = \frac{\pi \Omega_{\bar{n}}}{\lambda^2} k} \quad \left(\text{for } \Delta=0, t_R = \frac{2\pi \sqrt{\bar{n}+1}}{\lambda} k \cong \frac{2\pi \sqrt{\bar{n}}}{\lambda} k \right)$$

2. 4.6 $w(t) = \sum_n P_n \frac{1}{2} (e^{2i\lambda t \sqrt{n+1}} + e^{-2i\lambda t \sqrt{n+1}})$ from (4.183)

for a thermal state: $P_n = \frac{1}{(\bar{n}+1)} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n$

as before: $\sqrt{n+1} \cong \sqrt{\bar{n}+1} + \frac{n-\bar{n}}{2\sqrt{\bar{n}+1}} + \dots$

$$\cong A + Bn \quad \text{with} \quad \begin{cases} A = \sqrt{\bar{n}+1} - \frac{\bar{n}}{2\sqrt{\bar{n}+1}} \\ B = \frac{1}{2\sqrt{\bar{n}+1}} \end{cases}$$

$$\Rightarrow w(t) \cong \frac{1}{2(\bar{n}+1)} \sum_n \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \left[e^{2i\lambda t(A+Bn)} + e^{-2i\lambda t(A+Bn)} \right]$$

let $e^{2i\lambda t(A+Bn)} \rightarrow e^{i(A'+B'n)}$

$$\Rightarrow \sum_n \left(\frac{\bar{n}}{1+\bar{n}}\right)^n e^{i(A'+B'n)} = (1+\bar{n}) \frac{e^{iA'}}{1 - (e^{iB'} - 1)\bar{n}} \quad \text{with} \quad \begin{cases} A' = 2\lambda t A \\ B' = 2\lambda t B \end{cases}$$

$$1 - (e^{iB'} - 1)\bar{n} \cong 1 - iB'\bar{n} + \frac{B'^2}{2}\bar{n} + \dots$$

$$\cong 1 - iB'\bar{n} - \frac{B'^2\bar{n}^2}{2} + \frac{B'^2(\bar{n}^2 + \bar{n})}{2} + \dots$$

$$\cong e^{-iB'\bar{n} + B'^2(\bar{n}^2 + \bar{n})/2}$$

$$\Rightarrow \frac{e^{iA'}}{1 - (e^{iB'} - 1)\bar{n}} \cong e^{i(A'+B'\bar{n})} e^{-B'^2(\bar{n}^2 + \bar{n})/2}$$

$$\cong e^{2i\lambda t \sqrt{\bar{n}+1}} e^{-\frac{\lambda^2 t^2 \bar{n}}{2}}$$

$$\Rightarrow w(t) \cong \frac{1}{2(\bar{n}+1)} \left([\bar{n}+1] e^{-\frac{\lambda^2 t^2 \bar{n}}{2}} \right) \left(e^{2i\lambda t \sqrt{\bar{n}+1}} + e^{-2i\lambda t \sqrt{\bar{n}+1}} \right)$$

$$\cong \cos(2\lambda t \sqrt{\bar{n}+1}) e^{-\frac{\lambda^2 t^2 \bar{n}}{2}}$$

$$\Rightarrow \lambda_c = \lambda \sqrt{\frac{\bar{n}}{2}}$$

(for a thermal state $\Delta n \cong \bar{n}$, so following 4.128-4.129, we

and $\lambda_c (\Omega_n(\bar{n}+\bar{n}) - \Omega_n(\bar{n}-\bar{n})) \cong \lambda \sqrt{\bar{n}} \lambda_c \cong 1$

or $\lambda_c \propto \lambda \sqrt{\bar{n}}$, in agreement w/ the above!)

3.4.7 a) $H = \hbar(\hat{n} + \frac{1}{2})\omega + \hbar\lambda\hat{H}(b_+ + b_-)$ ^(ignore)

Clearly this only mixes: $|e\rangle_n \leftrightarrow |g\rangle_n$ (no change in n !)

So... $H_n = \hbar \begin{bmatrix} n\omega & \lambda\hbar \\ \lambda\hbar & n\omega \end{bmatrix}$

and... $\boxed{\begin{aligned} | \pm \rangle_n &= \frac{1}{\sqrt{2}} (|e\rangle \pm |g\rangle) |n\rangle \\ E_{\pm n} &= n(\omega \pm \lambda) \end{aligned}}$

b) $| \psi(t=0) \rangle = \sum_n C_n |g\rangle_n$
 $= \sum_n C_n \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle)$

$\Rightarrow | \psi(t) \rangle = \sum_n C_n e^{-in\omega t} \frac{1}{\sqrt{2}} \left(e^{-i\lambda t} |+\rangle_n - e^{+i\lambda t} |-\rangle_n \right)$
 $= \sum_n C_n e^{-in\omega t} \left[-\sin(n\lambda t) |e\rangle + \cos(n\lambda t) |g\rangle \right] |n\rangle$

$\Rightarrow P_e = \sum_n P_n \sin^2(n\lambda t)$

$\Rightarrow \omega(t) = 2P_e - 1 = \sum_n P_n [2\sin^2(n\lambda t) - 1] = -\sum_n P_n \cos(2n\lambda t)$

For a coherent state:

$P_n = e^{-\bar{n}} \frac{\bar{n}^n}{n!} \Rightarrow \omega(t) = -\sum_n e^{-\bar{n}} \frac{\bar{n}^n}{n} \cos(2n\lambda t)$

$\Rightarrow \omega(t) = -\frac{1}{2} e^{-\bar{n}} \left[\exp(\bar{n} e^{-2i\lambda t}) + \exp(\bar{n} e^{+2i\lambda t}) \right]$

(this same can be done in Mathematica if you don't know how...)

$\boxed{\omega(t) = -e^{-2\bar{n}\sin^2(\lambda t)} \cos[\bar{n}\sin(2\lambda t)]}$

Clearly, this function has period $T = \frac{\pi}{\lambda}$, and so the revivals are perfect! This is because all the terms are harmonic.

$$c) P_n = \frac{1}{n+1} \left(\frac{n}{n+1} \right)^n$$

$$\omega(t) = -\sum_n P_n \cos(2\lambda n t)$$

$$= - \frac{1}{2(n+1) - \frac{1+2n}{1+n-n\cos(2\lambda t)}} \\ 2n \sin^2(\lambda t)$$

(from Mathematica)

$$\boxed{\omega(t) = \frac{1 + 2n \sin^2(\lambda t)}{1 + 4n(n+1) \sin^2(\lambda t)}}$$

4. 4.8 a) $\frac{H}{\hbar} = \eta (a^2 \hat{b}_+ + a^{\dagger 2} \hat{b}_-) + \frac{1}{2} \omega_0 \hat{b}_3 + \hbar \omega$

This Hamiltonian mixes $|e\rangle|n\rangle \leftrightarrow |g\rangle|n+2\rangle$

$$\Leftrightarrow |\uparrow_1\rangle_n \leftrightarrow |\uparrow_2\rangle_n$$

$$\Rightarrow H_n = \hbar \begin{bmatrix} n\omega + \omega_0/2 & \eta \sqrt{(n+1)(n+2)} \\ \eta \sqrt{(n+1)(n+2)} & (n+2)\omega - \omega_0/2 \end{bmatrix}$$

The eigenstates are:

$$\frac{E_{\pm n}}{\hbar} = (n+1)\omega \pm \frac{\Omega_n(\Delta)}{2}$$

$$\text{where } \Omega_n \equiv \sqrt{4(n+1)(n+2)\eta^2 + \Delta^2}$$

$$\Delta \equiv \omega_0 - 2\omega$$

$$|\pm\rangle_n = \frac{1}{N_{\pm}} \left[\frac{\Delta \pm \Omega_n}{\Omega_n(0)} |\uparrow_1\rangle_n + |\uparrow_2\rangle_n \right]$$

This is identical to the Jaynes Cummings Hamiltonian, so...
(using 4.147 - 4.149)

$$|+\rangle_n = \cos(\Phi_n/2) |\uparrow_1\rangle_n + \sin(\Phi_n/2) |\uparrow_2\rangle_n$$

$$|-\rangle_n = -\sin(\Phi_n/2) |\uparrow_1\rangle_n + \cos(\Phi_n/2) |\uparrow_2\rangle_n$$

$$\Phi_n = \tan^{-1} \left(\frac{\Omega_n(0)}{\Delta} \right)$$

b) This is nearly identical to the first problem!

$$|\psi(t=0)\rangle = \sum_n C_n (\sin(\Phi_n/2) |+\rangle_n + \cos(\Phi_n/2) |-\rangle_n)$$

$$\Rightarrow \langle e | \psi(t) \rangle = \sum_n C_n |n\rangle e^{i n \omega t} \underbrace{\sin(\Phi_n/2) \cos(\Phi_n/2) \times (e^{-i \Omega_n t/2} - e^{+i \Omega_n t/2})}_{-2i \sin(\Omega_n t/2)}$$

$$\Rightarrow P_e(t) = \sum_n P_n \underbrace{\sin^2\left(\frac{\Phi_n}{2}\right) \cos^2\left(\frac{\Phi_n}{2}\right)}_{\frac{1}{4} \frac{\Omega_n^2 - \Delta^2}{\Omega_n^2}} 4 \sin^2(\Omega_n t/2)$$

$$\omega(t) = 2P_e(t) + 1 = \sum_n P_n \left(\frac{8 \Omega_n^2 (n+1)(n+2)}{\Omega_n^2} \sin^2\left(\frac{\Omega_n t}{2}\right) - 1 \right)$$

for a number state, $P_n = \delta_n$

for a coherent state, $P_n = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$

for $|n\rangle$, the inversion is periodic in Ω_n .

$$\text{from 4.137: } \underbrace{[\Omega(\bar{n}+1) - \Omega(\bar{n})]}_{\approx \frac{2\bar{n}^2(3+2\bar{n})}{\Omega_{\bar{n}}}} \tau_R = 2\pi k$$

$$\approx \frac{2\bar{n}^2(3+2\bar{n})}{\Omega_{\bar{n}}} \quad (\text{from expansion about } \bar{n})$$

$$\text{for } \bar{n} \gg 1, \Omega_{\bar{n}} \rightarrow 2\bar{n}\omega \Rightarrow \tau_r \approx \left(\frac{\pi}{\omega}\right) \bar{n} k$$

So the revivals are nearly periodic! See plot on next page.

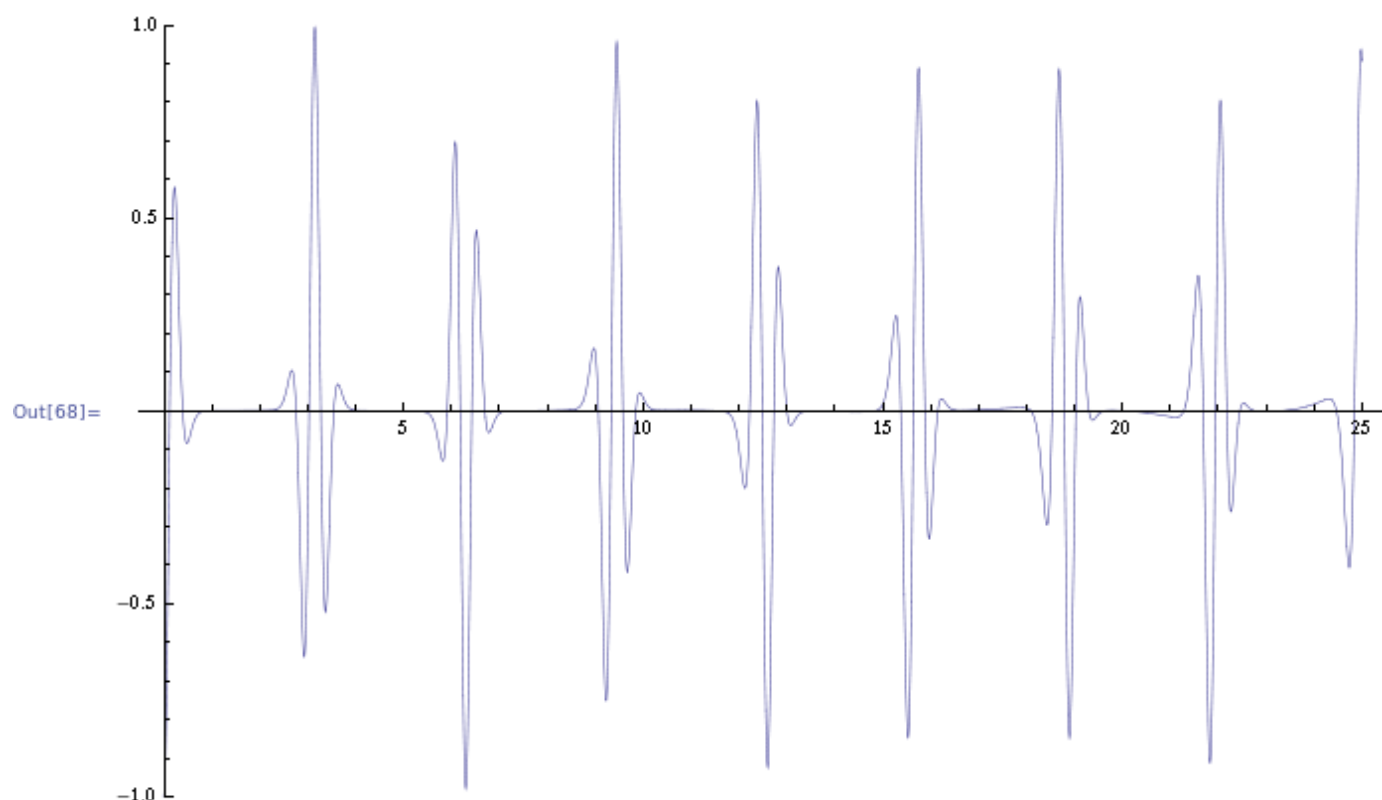
$$c) P_n = \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1}\right)^n \quad (\text{wctd above})$$

This obviously won't simplify. A plot is on the next page.


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In[67]:= W2 = Exp[-nb]  $\frac{nb^n}{n!} \left( \frac{8 \eta^2 (n+1) (n+2)}{\Omega^2} \sin[\Omega t / 2]^2 - 1 \right) /. \Omega \rightarrow \sqrt{\Delta^2 + 4 \eta^2 (n+1) (n+2)} ;$ 
Plot[Sum[W2 /. {η → 1, Δ → 0, nb → 5}, {n, 0, 20}], {t, 0, 25}, PlotRange → {-1, 1}]

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In[71]:= W3 =  $\frac{1}{nb+1} \left( \frac{nb}{nb+1} \right)^n \left( \frac{8 \eta^2 (n+1) (n+2)}{\Omega^2} \sin[\Omega t / 2]^2 - 1 \right) /. \Omega \rightarrow \sqrt{\Delta^2 + 4 \eta^2 (n+1) (n+2)} ;$ 
Plot[Sum[W3 /. {η → 1, Δ → 0, nb → 5}, {n, 0, 30}], {t, 0, 25}, PlotRange → {-1, 1}]

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