Physics CS 36 **Spring 2012**

Set #5 - for Thurs May 10

Read Ohanian	Ch. 5, Ch. 6 Sects 6.1-6.3
Read Feynman Vol. III	Ch. 1, Ch.2, Ch. 3

From Ohanian:

Ch. 5 Problems 33, 35, 36

1. Consider a dispersive medium for which the relationship between the angular frequency $\omega = 2\pi\nu$ and the wave number $k = \frac{2\pi}{\lambda}$ is given by:

$$\omega(k) = \omega_0(k^3/6k_0^3 - k^2/2k_0^2 + 7k/16k_0)$$

where ω_0 and k_0 are positive constants having units of frequency and wavenumber respectively.

a) Sketch a graph of ω versus k.

b) Find the phase and group velocities as functions of k. Interpret your results.

2. At $t = 0$, the wavefunction of a free electron is described by the wavepacket:

$$\psi(x,0) = \begin{cases} C\cos(k_0 x) & \text{if } |x| \leq L, \\ 0 & \text{if } |x| > L \end{cases}$$

a) Find the normalization constant C.

b) What is the probability of finding the electron in the region $0 \leq x \leq L$?

c) Write $\psi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk$. Find the distribution function $A(k)$. Sketch $A(k)$.

3. Spreading of a wavepacket (I)

We have seen that a wavepacket can be written in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega t)}dk,$$

where ω depends on k. Consider a Gaussian wavepacket $A(k) = A_0 e^{-(\Delta x_0)^2(k-k_0)^2}$ where Δx_0 is the initial spatial spread of the wavepacket. Calculate the wavefunction $\psi(x,t)$ from $A(k)$ as follows:

(i) Write $\omega(k) = \omega_0 + v_g(k-k_0) + \alpha(k-k_0)^2$, where $\omega_0 \equiv \omega(k_0)$ and we neglect higher-order terms in the Taylor expansion of $\omega(k)$ (note that v_g is the group velocity);

(ii) change the integration variable $\tilde{k} = k - k_0$;

(iii) take the factor $e^{i(k_0 x - \omega_0 t)}$ outside the integral;

(iv) make the substitution $(\Delta x)^2 = (\Delta x_0)^2 + i\alpha t$;

(v) complete the square of the quadratic function of $\tilde{k}$ (by adding and subtracting an appropriate term);

(vi) change the variable of integration to obtain an integral of the form $\int_{-\infty}^{\infty} e^{-a^2x^2} dx$ which has the value $\sqrt{\pi}/a$.

Your answer should be

$$\psi(x,t) = \frac{\pi^{\frac{1}{2}} A_0}{\Delta x}\, e^{\frac{-(x-v_g t)^2}{4(\Delta x)^2}}\, e^{i(k_0 x - \omega_0 t)}.$$

4. Spreading of a wavepacket (II)

a) Using the wavefunction in Problem 3, show that

$$|\psi(x,t)|^2 = \frac{\pi A_0^2}{(\Delta x_0)^2[1 + \alpha^2 t^2/(\Delta x_0)^4]^{\frac{1}{2}}}\; e^{-(x-v_g t)^2 / \{2(\Delta x_0)^2[1+\frac{\alpha^2 t^2}{(\Delta x_0)^4}]\}}$$

b) With what speed does the maximum of the function $|\psi(x,t)|^2$ move?

c) Sketch $|\psi(x,t)|^2$ at different times and discuss how its shape changes with time.

d) For non-relativistic matter waves, $\omega = \frac{\hbar k^2}{2m}$. What is the value of α in this case? What happens if α has the opposite sign? Does the wavefunction still spread? What if $\alpha = 0$?

5. Refer to the spreading wavepacket Problems 3 and 4. Use the $|\psi(x,t)|^2$ that you calculated and find the average of x:

$$x_{av} \equiv \langle x(t) \rangle = \frac{\int_{-\infty}^{\infty} x\ |\psi(x,t)|^2\ dx}{\int_{-\infty}^{\infty} |\psi(x,t)|^2\ dx}.$$

Interpret your result.

6. Find the standard deviation $\Delta x = \langle (x - \langle x \rangle) \rangle^{1/2}$. That is

$$(\Delta x)^2 = \frac{\int_{-\infty}^{\infty} (x - x_{av}(t))^2\ |\psi(x,t)|^2\ dx}{\int_{-\infty}^{\infty} |\psi(x,t)|^2\ dx},$$

where $|\psi(x,t)|^2$ is the same as the above.

Likewise, find $\Delta k = \langle (k - \langle k \rangle)^2 \rangle^{\frac{1}{2}}$, where $\langle k \rangle = \frac{\int_{-\infty}^{\infty} k|A(k)|^2 dk}{\int_{-\infty}^{\infty} |A(k)|^2 dk}$.

Show the exact uncertainty relation:

$$\Delta x(t) \Delta p = \frac{\hbar}{2} \sqrt{1 + \frac{\alpha^2 t^2}{(\Delta x(0))^4}} \geq \frac{\hbar}{2}$$

7. Imagine that we chop a continuous laser beam (assumed to be monochromatic at $\lambda_0 = 632.8$ nm) into 0.1 ns pulses using some sort of shutter. Compute the resultant uncertainty in the wavelength, $\Delta\lambda$ (linewidth). Compute the *bandwith* $\Delta\nu$ (uncertainty in the frequency). Compute the *coherence length* ($=c\Delta t$)