

## Set #9 - for Tues June 5

Read Ohanian Ch. 6 Sects 6.6-6.7Read Feynman Vol. III Ch. 19**From Ohanian:****Ch. 6** Problems 53, 62, 65, 66

1. A potential of the form  $V = V_0\delta(x)$ ,  $V_0 > 0$ , is introduced into the center of an infinite square well centered at the origin. Find the energy eigenfunctions (*unnormalized*) and show that for the odd parity solutions, the energy levels are the same as the corresponding energy levels for the infinite square well, and that for the even parity solutions the energy levels are determined by the equation

$$\tan(ka) = -\frac{k}{\gamma}, \quad \text{where } k^2 = \frac{2mE}{\hbar^2} \quad \text{and} \quad \gamma = \frac{mV_0}{\hbar^2}$$

2. Consider the definition of  $\hat{L}_x$ ,  $\hat{L}_y$  and  $\hat{L}_z$ , the components of the angular momentum operator, given in class:

$$\hat{L}_p = -i\hbar\epsilon_{pjk}x_j\frac{\partial}{\partial x_k}, \quad p, j, k = 1, 2, 3$$

Show that the commutators:

$$[\hat{L}_p, \hat{L}_j] \equiv \hat{L}_p\hat{L}_j - \hat{L}_j\hat{L}_p = i\hbar\epsilon_{pjk}\hat{L}_k \quad \text{and} \quad [\hat{L}^2, \hat{L}_j] = 0$$

3. The energy levels of the one-electron atom are

$$E_n = -\frac{Z^2e^4\mu}{32\pi^2\epsilon_0^2\hbar^2} \frac{1}{n^2}$$

For given  $n$  and any  $l \leq n - 1$ , the radial solutions can be written in the form

$$R = \frac{F(\rho)}{\rho} e^{-\rho/n}$$

With  $\rho = \frac{r}{a}$  and  $a = \frac{4\pi\epsilon_0\hbar^2}{Ze^2\mu}$ , then  $E_n = -\frac{\hbar^2}{2\mu a^2} \frac{1}{n^2}$ .  $F$  is the polynomial

$$F(\rho) = A \sum_{k=l+1}^n a_k \rho^k \quad \text{with} \quad a_{l+1} = 1$$

a) Starting with the equation satisfied by  $\chi = rR$  (derived in class) show that  $F(\rho)$  satisfies

$$\frac{d^2 F}{d\rho^2} - \frac{2}{n} \frac{dF}{d\rho} + \left[ \frac{2}{\rho} - \frac{l(l+1)}{\rho^2} \right] F = 0.$$

b) Derive the recursion relation,

$$a_{k+1} = 2 \frac{k/n - 1}{[k(k+1) - l(l+1)]} a_k$$

relating the successive coefficients in the polynomial. To get the polynomial  $F(\rho)$  make the power series expansion to terminate.

4. Use the recursion relation above to construct the radial polynomial for the case when  $n = 2$  and  $l = 0$ . Construct the normalized wavefunction  $\psi_{200}(r, \theta, \phi)$  and compare with Ohanian p. 205 for the case  $Z = 1, \mu \approx m_e$ . Determine the most probable distance between the electron and the nucleus for the state  $\psi_{200}(r, \theta, \phi)$  and compare with Bohr's orbit model. Calculate the expectation values of  $r$  and  $V(r)$ .

**Extra Credit 1:** Consider again the potential  $V(x) = -V_0\delta(x)$ . Now assume that a beam of particles of mass  $m$  is incident from the left with fixed momenta. Compute the reflection and transmission coefficients. Express your answer in terms of  $p$ , the momentum of the incident particles, and  $V_0 > 0$ .

**Extra Credit 2:** Let  $\hat{D} = \frac{d}{dx}$  and  $\hat{A} = \hat{D}^4 - 2x^2\hat{D}^2 - 4x\hat{D} + x^4 - 2$ ,

a) Show that  $\hat{A} = (\hat{D}^2 - x^2)^2$

b) Is  $\hat{A}$  a Hermitian operator?

c) Find the eigenvalues and eigenfunctions of  $\hat{A}$ .

*N.B. For any operator  $\hat{O}$  if  $\hat{O}\psi(x) = \lambda\psi(x)$  then  $F(\hat{O})\psi = F(\lambda)\psi$ , where  $F(\hat{O})$  is a function of the linear operator  $\hat{O}$ .*

**Extra Credit 3:** Consider the so-called *generating function* for Hermite polynomials:

$$S(s, \rho) = e^{\rho^2 - (s-\rho)^2} \quad (1)$$

a) Show that  $S(s, \rho)$  satisfies the equation:

$$\frac{\partial^2 S}{\partial \rho^2} - 2\rho \frac{\partial S}{\partial \rho} + 2s \frac{\partial S}{\partial s} = 0.$$

b) Now write  $S$  in the form of a power series in  $s$ :

$$S(\rho, s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} f_n(\rho) \quad (2)$$

Substitute this expansion into the equation of part (a) and show that the  $f_n(\rho)$  satisfy the Hermite differential equation:

$$f_n'' - 2\rho f_n' + 2n f_n = 0. \quad (3)$$

The polynomial solutions of (3) such that when substituted into (2) will give  $S$  as given by (1), are the Hermite polynomials  $H_n(\rho)$ . Thus

$$S(s, \rho) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\rho). \quad (4)$$

c) Use (1) to show that

$$\lim_{s \rightarrow 0} \frac{d^n S(s, \rho)}{ds^n} = e^{\rho^2} (-1)^n \frac{d^n}{d\rho^n} e^{-\rho^2},$$

and thus  $H_n(\rho) = e^{\rho^2} (-1)^n \frac{d^n}{d\rho^n} e^{-\rho^2}$  is the general expression for a Hermite polynomial of degree  $n$ .

d) Write down explicitly the expressions for  $H_0$ ,  $H_1$ ,  $H_2$  and  $H_3$ .